A Proportionally Geometric Cantor Set

Anastasia Clements

EPaDel Section Meeting - Undergraduate Speaker Session

Fall 2024

Contents

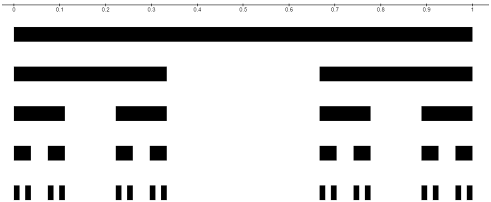


Figure: Ternary Cantor Set up to C_4

Introduction

The Cantor Set in Recursion

The right and left endpoints for a given sub-interval occuring at C_n of the middle-thirds Cantor set can be defined recursively as follows:

$$R(a, b, n) = \frac{a + b - \frac{1}{3}|a - b|}{2}$$

$$L(a, b, n) = \frac{a + b + \frac{1}{3}|a - b|}{2}$$

Where a, b are the endpoints of a prior subinterval (where the cut is occuring) in C_{n-1} and n is the current cut at C_n.

Notice that the size of the cut interval is proportional to the measure of the previous interval $|a - b|$.

The Cantor Set in Recursion

The right and left endpoints for a given sub-interval occuring at C_n of the middle-thirds Cantor set can be defined recursively as follows:

$$R(a, b, n) = \frac{a + b - \frac{1}{3}|a - b|}{2}$$

$$L(a, b, n) = \frac{a + b + \frac{1}{3}|a - b|}{2}$$

Where a, b are the endpoints of a prior subinterval (where the cut is occuring) in C_{n-1} and n is the current cut at C_n.

Notice that the size of the cut interval is proportional to the measure of the previous interval $|a - b|$.

The Cantor Set in Recursion

The right and left endpoints for a given sub-interval occuring at C_n of the middle-thirds Cantor set can be defined recursively as follows:

$$R(a, b, n) = \frac{a + b - \frac{1}{3}|a - b|}{2}$$

$$L(a, b, n) = \frac{a + b + \frac{1}{3}|a - b|}{2}$$

Where a, b are the endpoints of a prior subinterval (where the cut is occuring) in C_{n-1} and n is the current cut at C_n.

Notice that the size of the cut interval is proportional to the measure of the previous interval $|a - b|$.

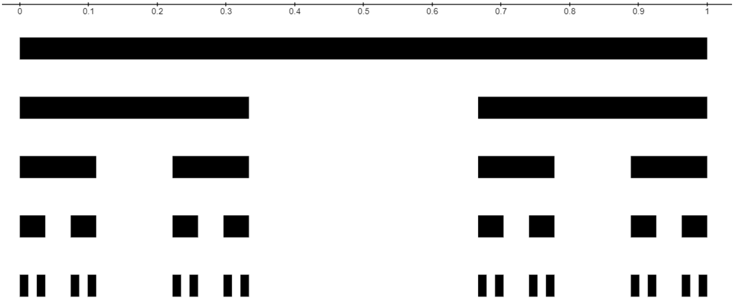


Figure: Ternary Cantor Set up to C_4

Smith-Volterra-Cantor Set

The Smith-Volterra-Cantor set, also known colloquially as the fat Cantor set, is defined by taking geometrically decreasing cuts from the unit interval.

The standard Smith-Volterra-Cantor set is defined as cutting $\left(\frac{1}{4}\right)^n$ from each sub-interval in C_{n-1}.

This is a unique way of cutting the unit interval, as it leaves a perfect, no-where dense set (like the middle-thirds Cantor set); however, the standard Smith-Volterra-Cantor set retains a measure of $\frac{1}{2}$.

Smith-Volterra-Cantor Set

The Smith-Volterra-Cantor set, also known colloquially as the fat Cantor set, is defined by taking geometrically decreasing cuts from the unit interval.

The standard Smith-Volterra-Cantor set is defined as cutting $\left(\frac{1}{4}\right)^n$ from each sub-interval in C_{n-1}.

This is a unique way of cutting the unit interval, as it leaves a perfect, no-where dense set (like the middle-thirds Cantor set); however, the standard Smith-Volterra-Cantor set retains a measure of $\frac{1}{2}$.

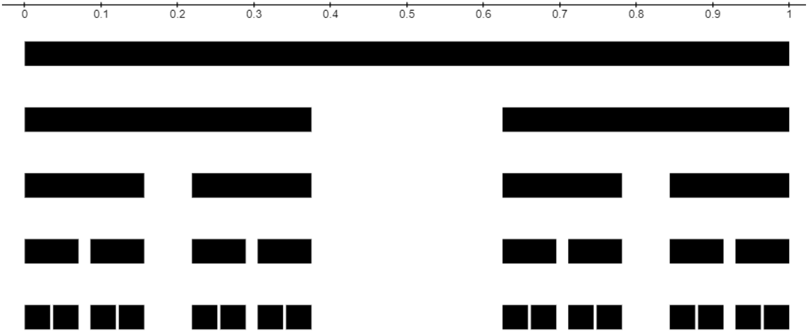


Figure: Smith-Volterra-Cantor Set up to C_4

Similar to the middle-thirds Cantor set, the endpoints in C_n of the Smith-Volterra-Cantor set can be naturally defined recursively as follows:

$$R(a, b, n) = \frac{a + b - \left(\frac{1}{4}\right)^n}{2}$$

$$L(a, b, n) = \frac{a + b + \left(\frac{1}{4}\right)^n}{2}$$

Where a, b are the endpoints of a prior sub-interval in C_{n-1} and n is the current cut at C_n.

Notice that this recursion is lacking a key feature that was included in the middle-thirds Cantor set: proportionality!

Similar to the middle-thirds Cantor set, the endpoints in C_n of the Smith-Volterra-Cantor set can be naturally defined recursively as follows:

$$R(a,b,n) = \frac{a+b-\left(\frac{1}{4}\right)^n}{2}$$

$$L(a,b,n) = \frac{a+b+\left(\frac{1}{4}\right)^n}{2}$$

Where a, b are the endpoints of a prior sub-interval in C_{n-1} and n is the current cut at C_n.

Notice that this recursion is lacking a key feature that was included in the middle-thirds Cantor set: proportionality!

Interpretation

It is from this modification of the standard Smith-Volterra-Cantor set that the definition of a Proportionally Geometric Cantor set comes from.

Recursively, the endpoints of a sub-interval in C_n of our Proportionally Geometric Cantor set is defined as one would expect:

$$R(a, b, n) = \frac{a + b - \left(\frac{1}{4}\right)^n |a - b|}{2}$$

$$L(a, b, n) = \frac{a + b + \left(\frac{1}{4}\right)^n |a - b|}{2}$$

Where a, b are distinct endpoints of a sub-interval of C_{n-1}.

Interpretation

It is from this modification of the standard Smith-Volterra-Cantor set that the definition of a Proportionally Geometric Cantor set comes from.

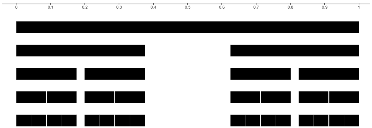
Recursively, the endpoints of a sub-interval in C_n of our Proportionally Geometric Cantor set is defined as one would expect:

$$R(a, b, n) = \frac{a + b - \left(\frac{1}{4}\right)^n |a - b|}{2}$$

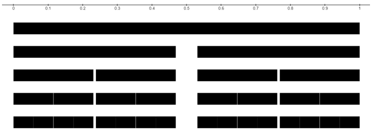
$$L(a, b, n) = \frac{a + b + \left(\frac{1}{4}\right)^n |a - b|}{2}$$

Where a, b are distinct endpoints of a sub-interval of C_{n-1}.

Note that this Proportionally Geometric Cantor set preserves an essence of self-similarity, as each sub-interval is itself almost a Proportionally Geometric Cantor set.
The reason Proportionally Geometric Cantor sets are not self-similar is that the starting cut is now shifted ahead n "steps" and scaled down, which is not an exact copy of the original set.

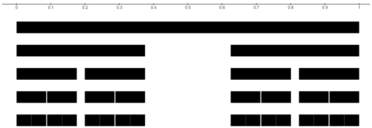


(a) PG Cantor Set up to C_4

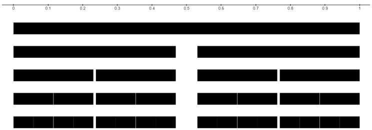


(b) C_1 sub-interval scaled by $\frac{1}{l(1)}$

From these recursive definitions, explicit equations for total measure, sub-interval length, cut length at a given C_n, and endpoints location can be derived.



(a) PG Cantor Set up to C_4



(b) C_1 sub-interval scaled by $\frac{1}{I(1)}$

From these recursive definitions, explicit equations for total measure, sub-interval length, cut length at a given C_n, and endpoints location can be derived.

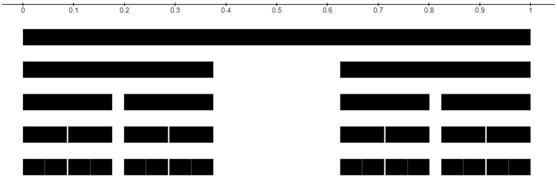


Figure: Proportionally Geometric Cantor Set up to C_4

A Proportionally Geometric Cantor Set

Explicit Formula for Length and Measure

By taking our recursive equation $R(a, b, n)$ with $a = 0$, we can find an explicit equation for the length of a sub-interval. In the interest of standardization, all formulas will be written in terms of the total measure at some C_n.

Note this is derived from the sub-interval equation, and not vice-versa.

The measure function $d(n)$ for a Proportionally Geometric Cantor set (with this construction) at some C_n is the following partial product:

$$d(n) = \prod_{i=1}^{n} \left(1 - \left(\frac{1}{4}\right)^i\right).$$

Explicit Formula for Length and Measure

By taking our recursive equation $R(a, b, n)$ with $a = 0$, we can find an explicit equation for the length of a sub-interval. In the interest of standardization, all formulas will be written in terms of the total measure at some C_n.

Note this is derived from the sub-interval equation, and not vice-versa.

The measure function $d(n)$ for a Proportionally Geometric Cantor set (with this construction) at some C_n is the following partial product:

$$d(n) = \prod_{i=1}^{n} \left(1 - \left(\frac{1}{4}\right)^i\right).$$

From the equation for the measure at C_n, we can determine explicitly the length of a sub-interval at some C_n (denoted $I(n)$) as well as the individual size of the cut(s) performed at that C_n (denoted $J(n)$).

$$I(n) = \frac{1}{2^n} d(n)$$

$$J(n) = \frac{1}{2^{n-1}(4^n - 1)} d(n)$$

These equations are tremendously useful in explicitly determining the endpoints at a given C_n.

From the equation for the measure at C_n, we can determine explicitly the length of a sub-interval at some C_n (denoted $I(n)$) as well as the individual size of the cut(s) performed at that C_n (denoted $J(n)$).

$$I(n) = \frac{1}{2^n} d(n)$$

$$J(n) = \frac{1}{2^{n-1}(4^n - 1)} d(n)$$

These equations are tremendously useful in explicitly determining the endpoints at a given C_n.

Measure of PG Cantor Set

Let us examine the $\lim_{n\to\infty} d(n)$.

Utilizing our product equation for $d(n)$, we have the following:

$$\lim_{n\to\infty} d(n) = \prod_{i=1}^{\infty} \left(1 - \left(\frac{1}{4}\right)^i\right)$$

This infinite product converges, call it d (and is represented by the q-Pochhammer Symbol $\left(\frac{1}{4};\frac{1}{4}\right)_\infty$).

$$d = \left(\frac{1}{4}\right)_\infty \approx 0.68853753712\ldots$$

Measure of PG Cantor Set

Let us examine the $\lim_{n\to\infty} d(n)$.
Utilizing our product equation for $d(n)$, we have the following:

$$\lim_{n\to\infty} d(n) = \prod_{i=1}^{\infty}\left(1-\left(\frac{1}{4}\right)^{i}\right)$$

This infinite product converges, call it d (and is represented by the q-Pochhammer Symbol $\left(\frac{1}{4};\frac{1}{4}\right)_{\infty}$).

$$d = \left(\frac{1}{4}\right)_{\infty} \approx 0.68853753712\ldots$$

Measure of PG Cantor Set

Let us examine the $\lim_{n\to\infty} d(n)$.
Utilizing our product equation for $d(n)$, we have the following:

$$\lim_{n\to\infty} d(n) = \prod_{i=1}^{\infty} \left(1 - \left(\frac{1}{4}\right)^i\right)$$

This infinite product converges, call it d (and is represented by the q-Pochhammer Symbol $\left(\frac{1}{4}; \frac{1}{4}\right)_\infty$).

$$d = \left(\frac{1}{4}\right)_\infty \approx 0.68853753712\ldots$$

Measure of PG Cantor Set

Let us examine the $\lim_{n\to\infty} d(n)$.
Utilizing our product equation for $d(n)$, we have the following:

$$\lim_{n\to\infty} d(n) = \prod_{i=1}^{\infty}\left(1-\left(\frac{1}{4}\right)^{i}\right)$$

This infinite product converges, call it d (and is represented by the q-Pochhammer Symbol $\left(\frac{1}{4};\frac{1}{4}\right)_{\infty}$).

$$d = \left(\frac{1}{4}\right)_{\infty} \approx 0.68853753712\ldots$$

Known Properties

Currently we have verified the following properties of our Proportionally Geometric Cantor set:

- C is perfect (closed and every point in C is a limit point of C).
- C is no-where dense.
- C is uncountable.
- C has finite, non-zero measure.

Open Questions

- What is the Hausdorff Dimension of a Proportionally Geometric Cantor set?
- What do higher dimensional constructions of Proportionally Geometric Cantor sets look like?

Thank you for attending!

Bonus: Proof of Measure function

Proof: We shall prove by induction that the measure function

$$d(n) = \prod_{i=1}^{n} \left(1 - \left(\frac{1}{4}\right)^i\right)$$

for all integers $n \geq 1$.

We begin with the observation that at C_n there are exactly 2^n sub-intervals.

We define $r(n)$ to be the first (left-most) right endpoint of C_n so that $r(0) = 1$.

$r(n)$ can be defined recursively for $n \geq 1$:

$$r(n) = R(0, r(n-1), n)$$

Bonus: Proof of Measure function

Proof: We shall prove by induction that the measure function

$$d(n) = \prod_{i=1}^{n} \left(1 - \left(\frac{1}{4}\right)^{i}\right)$$

for all integers $n \geq 1$.

We begin with the observation that at C_n there are exactly 2^n sub-intervals.

We define $r(n)$ to be the first (left-most) right endpoint of C_n so that $r(0) = 1$.

$r(n)$ can be defined recursively for $n \geq 1$:

$$r(n) = R(0, r(n-1), n)$$

Bonus: Proof of Measure function

Proof: We shall prove by induction that the measure function

$$d(n) = \prod_{i=1}^{n} \left(1 - \left(\frac{1}{4}\right)^{i}\right)$$

for all integers $n \geq 1$.

We begin with the observation that at C_n there are exactly 2^n sub-intervals.

We define $r(n)$ to be the first (left-most) right endpoint of C_n so that $r(0) = 1$.

$r(n)$ can be defined recursively for $n \geq 1$:

$$r(n) = R(0, r(n-1), n)$$

Proof of Measure function

Since each sub-interval is equal in length, we have the following relationship:

$$d(n) = 2^n \cdot r(n).$$

This gives us the following:

$$\begin{aligned} d(n) &= 2^n \cdot r(n) \\ &= 2^n \cdot R(0, r(n-1), n) \\ &= 2^n \cdot \frac{0 + r(n-1) - (\frac{1}{4})^n |0 - r(n-1)|}{2} \\ &= 2^{n-1} \cdot r(n-1) \left(1 - \left(\frac{1}{4}\right)^n\right) \\ &= d(n-1) \left(1 - \left(\frac{1}{4}\right)^n\right) \end{aligned}$$

From here we can begin induction.

Proof of Measure function

Since each sub-interval is equal in length, we have the following relationship:

$$d(n) = 2^n \cdot r(n).$$

This gives us the following:

$$\begin{aligned}
d(n) &= 2^n \cdot r(n) \\
&= 2^n \cdot R(0, r(n-1), n) \\
&= 2^n \cdot \frac{0 + r(n-1) - (\frac{1}{4})^n |0 - r(n-1)|}{2} \\
&= 2^{n-1} \cdot r(n-1) \left(1 - \left(\frac{1}{4}\right)^n\right) \\
&= d(n-1) \left(1 - \left(\frac{1}{4}\right)^n\right)
\end{aligned}$$

From here we can begin induction.

Proof of Measure function

Since each sub-interval is equal in length, we have the following relationship:

$$d(n) = 2^n \cdot r(n).$$

This gives us the following:

$$\begin{aligned} d(n) &= 2^n \cdot r(n) \\ &= 2^n \cdot R(0, r(n-1), n) \\ &= 2^n \cdot \frac{0 + r(n-1) - (\frac{1}{4})^n |0 - r(n-1)|}{2} \\ &= 2^{n-1} \cdot r(n-1) \left(1 - \left(\frac{1}{4}\right)^n\right) \\ &= d(n-1) \left(1 - \left(\frac{1}{4}\right)^n\right) \end{aligned}$$

From here we can begin induction.

Proof of Measure function

Base Case at $n = 1$:

$$\begin{aligned} d(1) &= 2^0 \cdot r(0)\left(1 - \left(\frac{1}{4}\right)^1\right) \\ &= 1 \cdot 1 \cdot \left(1 - \frac{1}{4}\right) \\ &= \frac{3}{4} \\ &= \prod_{i=1}^{1}\left(1 - \frac{1}{4}\right) \end{aligned}$$

thus proving our statement holds for $n = 1$.

Proof of Measure function

Base Case at $n = 1$:

$$\begin{aligned} d(1) &= 2^0 \cdot r(0)\left(1 - \left(\frac{1}{4}\right)^1\right) \\ &= 1 \cdot 1 \cdot \left(1 - \frac{1}{4}\right) \\ &= \frac{3}{4} \\ &= \prod_{i=1}^{1}\left(1 - \frac{1}{4}\right) \end{aligned}$$

thus proving our statement holds for $n = 1$.

Inductive Step: We assume that $d(n) = \prod_{i=1}^{n}(1 - (\frac{1}{4})^i)$ holds for some values of n (as shown in the base case $n = 1$).

We will show that $d(n+1)$ holds:

$$\begin{aligned} d(n+1) &= 2^n \cdot r(n)\left(1 - \left(\frac{1}{4}\right)^{n+1}\right) \\ &= d(n)\left(1 - \left(\frac{1}{4}\right)^{n+1}\right) \\ &= \left(\prod_{i=1}^{n}\left(1 - \left(\frac{1}{4}\right)^{i}\right)\right)\left(1 - \left(\frac{1}{4}\right)^{n+1}\right) \\ &= \prod_{i=1}^{n+1}\left(1 - \left(\frac{1}{4}\right)^{i}\right) \end{aligned}$$

as required. □

Inductive Step: We assume that $d(n) = \prod_{i=1}^{n}(1 - (\frac{1}{4})^i)$ holds for some values of n (as shown in the base case $n = 1$).

We will show that $d(n+1)$ holds:

$$
\begin{aligned}
d(n+1) &= 2^n \cdot r(n)\left(1 - \left(\frac{1}{4}\right)^{n+1}\right) \\
&= d(n)\left(1 - \left(\frac{1}{4}\right)^{n+1}\right) \\
&= \left(\prod_{i=1}^{n}\left(1 - \left(\frac{1}{4}\right)^{i}\right)\right)\left(1 - \left(\frac{1}{4}\right)^{n+1}\right) \\
&= \prod_{i=1}^{n+1}\left(1 - \left(\frac{1}{4}\right)^{i}\right)
\end{aligned}
$$

as required. $\square$