Math Dept Colloquium, West Chester University

One, Two, Many (or a dozen reasons why mathematics isn't as easy as 1,2,3)

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Abstract

Are there really primitive tribes whose system of counting goes:

"One, Two, Many,. . . "

indicating that from three on it's more or less a blur?

Maybe we modern humans are such a tribe.

Despite the sophistication we see in ourselves compared with our less advanced ancestors from times long past, it's surprising how little progress we've made in addressing some basic problems in 3D or beyond, or when solving seemingly simple equations in ≥ 3 variables.

While our accomplishments include astonishing mastery of some ABCs, such as Air (flight, weather prediction), Biology (medical breakthroughs, DNA) and Communications (phone, video, email), we often struggle to get past 1, 2, 3, in other domains.

Or sometimes even to get to 3.

We'll survey a dozen fun topics in shapes and numbers and patterns whose basics and generalisations can be explored with little mathematical background, and which speedily lead to "what if?" questions ranging from easy to tricky to "we just don't know."

Fruit, cakes, doughnuts, bagels, coins, boxes, cubes, primes, squares, and sums involving powers will all make appearances.

Once or twice we will stray into deeper waters and touch on more sophisticated topics.

(At the end we mention some advanced forays into modern math.)

Columnist Martin Gardner (1914-2010), whose legacy includes 100+ non-fiction books (40 of them on math/puzzles), knew all too well that playful queries can both excite students about maths AND lead to real research at the frontiers of the subject.

Today, you will have satisfying *Aha!* moments, there will be room for innovative ideas, and million dollar prizes will be discussed.

Some of the results we'll mention are quite recent.

Recall the wise words of Bob Crease (*Physics World*, Oct 2014):

> *"Googling is not the Gardner way. The Gardner way is to ignite your fascination so that you experience the pleasure of finding the answer yourself."*

In that spirit, many identifying names have been redacted below!

GOAL: stimulate your curiosity!

0. Close to Home (Familial Territory)

Vertical search:

Most of us know the names and faces of our parents, and have spent time with them.

What about our grandparents?

Or our great grandparents?

Horizontal search:

Most of us know the names and faces of our siblings, and have spent time with them.

What about our first cousins? Do you know many you have?

Or our second or third cousins? Do you know many you have?

1. Fruit Fandango

Imagine that you are shopping in a *Less Than a Dollar Store*, where every item they sell costs less than \$1. Perhaps some cost 10c or 24c, others, 59c or 70c, a few might cost 95c or 99c.

There are two unmarked items you particularly wish to know the prices of, packages of frozen Apples and tins of Bananas.

You ask for help and are told that you can scan any item or combination of items you wish and the giant scanner will give the total cost.

For instance, scanning 3 packages of frozen Apples and 2 tins of Bananas yields the total cost, let's say it's \$1.32.

Alas, that's not enough information to work out the individual prices.

Scanning just one package of Apples, and then scanning a single tin of Bananas would do the trick of course.

However, the machine is on the blink, and after a single use it will shut down for hours. Using it twice simply isn't an option.

> *Q1. With a single scan, can you deduce the cost of both the apples and the bananas?*

Now imagine that you are interested in the unpriced jars of pickled Clementines as well.

With a single scan, can you deduce the cost of all 3 types of fruit: Apples, Bananas and Clementines?

2. Doughnut Division

Continuing with a food theme for now, a doughnut can be sliced in various ways with straight knife cuts. Clearly the largest number of pieces that be generated with a single planar cut is 2. The case of 2 planar cuts is not hard.

> *Q2. (Gardner) What is the largest number of pieces that be generated with 3 planar cuts of a doughnut?*

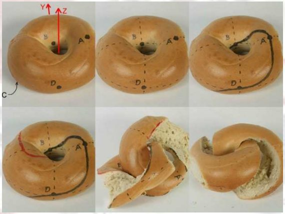
These questions can be asked anew if one is allowed to re-arrange the pieces after each cut, and the answers change quite a bit.

Let's not sugarcoat things: doughnuts tend to crumble when cut.

3. Bagel Bedazzler

Bagels are made of tougher stuff, and to a mathematician they are the same as doughnuts.

There is an exotic form of bagel cutting with a knife that yields the surprising result shown, namely 2 interlocking bagel halves. (See Georga Hart's "Mathematically Correct Breakfast" video online.)



Q3. Can this be modified to yield 3 (or more) parts?

The interlocking bagel halves trick is based on a familiar puzzle involving cutting loops of paper whose ends are taped together.

4. Cake Cutting

Imagine an ideal rectangular cake with chocolate icing uniformly spread on the top and sides. It's easy to cut it with a knife and distribute the results to 2 people who at least theoretically get equal amounts of cake and equal amounts of icing.

Q4. How can this be achieved for 3 (or more) people, using only straight knife cuts?

Assume we can cut $\frac{1}{3}$ of the way (or likewise) along any side, parallel to the perpendicular sides.

A much harder version concerns cakes which are not regular in shape or uniform in composition, either inside or in their icing, and where the preferences of the participants (for cake, fruit, cream, icing, etc) may vary. The solution here for 2 people has been known since biblical times (chapter 13, *Book of Genesis*).

In the 1940s it was figured out (Hugo Steinhaus) how to guarantee that each of n people gets a piece that they value as worth at least $1/n$, but that doesn't guarantee that they don't value someone else's piece more highly.

"Envy-free" cake cutting was introduced in the 1950s (George Gamow and Marvin Stern), and a solution for 3 people was developed around 1960 (Selfridge-Conway). That version was extended (with difficulty) to 4 or 5 people.

In 2016, a big breakthrough occurred for general n (Hariz Aziz and Simon Mackenzie): they showed that envy-free cake-cutting can be done in bounded time for 4 or more people.

5. Kerry Coin Conundrum

Imagine a country with only two types of coins, say, 7 and 5 dingle coins. You can't pay exactly for something that costs 1, 2, 3, 4, 6, 8, 9 or 11 dingles.

The same applies to something that costs 23 dingles, as well as some other values we skipped over.

From a certain point on, however, all is well.

Certainly, for anything costing 40 dingles or more, there is some combination of 5 and 7 dingle coins that will do the trick (why?).

It's key that 5 and 7 share no common factors. Hence every whole number is "an integer linear combination" of 5 and 7. The trick is to only use non-negative multiples of each value.

It's natural to ask what the crossover point is, in other words, the first whole number n so that everything costing n or more dingles can be paid for exactly?

From what we have claimed, it's between 24 and 40 inclusive. Its exact value is found with a little experimentation.

There is a simple formula (known since 1882) for the crossover value in general, regardless of the specific relatively prime values x and y of the coins in use.

We encourage you to discover this for yourself. It's one of many fascinating interactions between addition and multiplication.

However, if there are 3 (or more) coin values, sharing no common factors, suddenly things get much harder.

(Sometimes this is posed in terms of postage stamps.)

Q5. (Frobenius Problem) What is the formula that gives the crossover value if 3 coin types are in use, namely the first whole number n to that everything costing n or more can be paid for exactly with the 3 available coin types?

Amazingly, there is no known general formula in terms of the values of the available coins.

This "simple" question concerns representations of positive whole numbers as non-negative whole number combinations of positive coordinates (x, y, z) in a 3D integer lattice.

How hard can it be?

It's "really hard" in a specific technical sense.

Of course, for any specific 3 (or more) coin denominations, the crossover value can be found by routine trial and error.

Wikipedia:

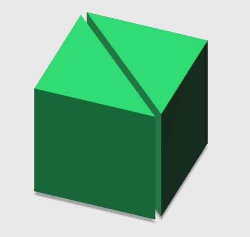
> *For any fixed number of coin denominations, there is an algorithm computing the Frobenius number in polynomial time (in the logarithms of the coin denominations forming an input). No known algorithm is polynomial time in the number of coin denominations, and the general problem, where the number of coin denominations may be as large as desired, is NP-hard.*

6. Baffling Box

Bisecting a rectangle into identically shaped triangles can be done with a simple diagonal cut.

Q6. What is the corresponding fact for a 3D box?

Here's a half-hearted effort in the case of a cube:



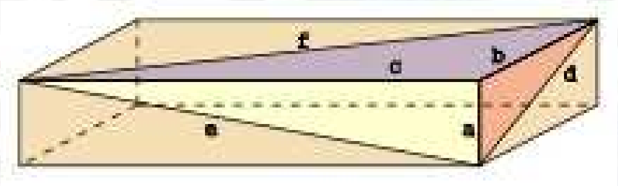
We can do much better, as several ancient cultures figured out.

7. Perfect Package

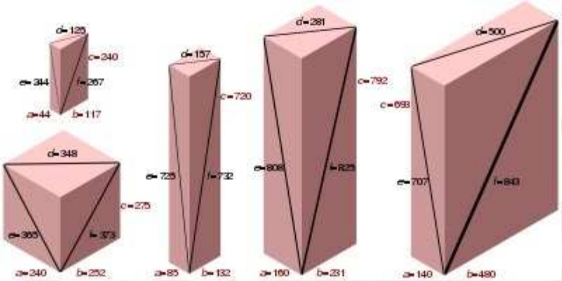
Consider traditional "Pythagorean" triples of whole numbers such as (3,4,5) or (5,12,13). The smaller numbers can be viewed as the lengths of the sides a rectangle whose (equal) diagonals—the largest numbers—are also whole numbers.

We can try to go up a dimension and ask:

> *Q7. Is there a 3D rectangular box whose sides, surface diagonals AND space diagonals are whole numbers?*



A $44 \times 117 \times 240$ box is superficially good. The same applies to a $240 \times 252 \times 275$ box, and the other "small" examples shown here:



These are called Euler bricks. The bottom left box here is the most cubelike. No such (Euler) cube can exist. (Why not?)

The question remains of whether a perfect Euler brick exists, i.e., one with whole number space diagonals too?

8. Series Subtlety

In 1734, Leonhard Euler solved a problem posed in 1650, gaining immediate fame as a corollary:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} + \ldots = \frac{\pi^2}{6}$$

We now know that this is also the value of $\int_0^1 \int_0^1 \frac{1}{1-xy}\, dx\, dy$.

Soon thereafter, he established that

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots + \frac{1}{n^4} + \ldots = \frac{\pi^4}{90}$$

Yes, it's the value of $\int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{1}{1-xyzw}\, dx\, dy\, dz\, dw$.

The obvious question to ask is,

Q8. What is the value of

$$\sum_{k=1}^{\infty} \frac{1}{k^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \ldots + \frac{1}{n^3} + \ldots \quad ?$$

Is it $\frac{\pi^3}{M}$ for some nice number M? We doubt it.

It wasn't until 1978 that this sum was even proved to be irrational (by Roger Apéry; and it's noted as such on his tombstone).

Presumably it's transcendental–i.e., doesn't satisfy any polynomial equation with integer coefficients (like $\sqrt[5]{3.14159 + \sqrt{6}}$ does).

Certainly, it's $\int_0^1 \int_0^1 \int_0^1 \frac{1}{1-xyz}\ dx\ dy\ dz$.

These are not idle curiosities, such numbers turn up in physics and elsewhere.

Moreover, the associated Riemann zeta function

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \ldots + \frac{1}{n^s} + \ldots$$

has received considerable attention since the 1800s. A key question about it (The Riemann Hypothesis, 1859) has a $1 million prize associated with it (via the Clay Mathematics Institute).

In addition, $\zeta(s) = \prod_{p \text{ prime}} \frac{p^s}{p^s - 1}$ provides another surprising connection between addition and multiplication.

The probability that s randomly chosen positive whole numbers are relatively prime is $\frac{1}{\zeta(s)}$.

9. 4D Duck

Many mathematical observations say something like:

If it's a duck, then it walks and talks like a duck.

Sometimes, the converse is also true, though harder to prove:

If it walks and talks like a duck, then it's a duck.

Spheres exist in all dimensions, and just to keep us on our toes an ordinary 3D sphere's surface is called a 2-sphere. The boundary of a 2D sphere is a 1-sphere, i.e., a circle's circumference.

We can extend this to higher dimensions, e.g., a 4D sphere is called a 3-sphere, namely the set $\{(x, y, z, w) |\ x^2 + y^2 + z^2 + w^2 = r^2\}$.

Q9. If it walks and talks like a 4D sphere, is it one?

Conjectured by Poincaré in 1904, and finally proved by Grigori Perelman (2002–2003). More formally, "Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere."

(Of course, 4D mathematics is a vast and fascinating universe unto itself, but we discuss this no more today.)

Perelman was offered a Fields medal, and in turn, \$1 million (Clay Mathematics Institute's Millennium Prize).

He turned both down.

Wikipedia says:

> *"he believed his contribution in proving the Poincaré conjecture was no greater than Hamilton's."*

Richard Hamilton's crucial work (on Ricci curvature) was done in the early 1980s. This was while he was on the faculty at Cornell, a fact seemingly overlooked today. (Likewise, Walter Feit was at Cornell in the early 1960s when he made his contributions to the landmark Feit-Thompson Theorem in group theory.)

10. Primes

(Peter Dirichlet, 1837) If a and b share no factors, then there are infinitely many primes of the form $f(n) = a + bn$.

So there are infinitely many primes of the form $1 + 2n$, $2 + 15n$, $9 + 35n$, and even $2022 + 2021n$ or $2021 + 2022n$.

A natural generalisation to wonder about is:

> *Q10. If a, b and c share no factors, are there infinitely many primes of the form $f(n) = a + bn + cn^2$?*

For this 1857 question (Viktor Bunyakovsky) to have a positive answer we need $f(x)$ irreducible and $f(1), f(2), f(3), \ldots$ sharing no factors.

We haven't proved this in a single special case, not even $1 + n^2$.

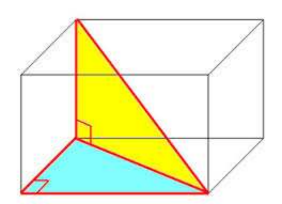
A further generalisation implies the Twin Prime conjecture.

11. SOS (and other causes for panic)

The classical Pythagorean equation $x^2 + y^2 = z^2$ suggests generalisations which also involve squares, such as

Does $x^2 + y^2 + z^2 = w^2$ *have interesting solutions?*

This relates to rectangular boxes: w is the space diagonal of the box with dimensions x, y, z.

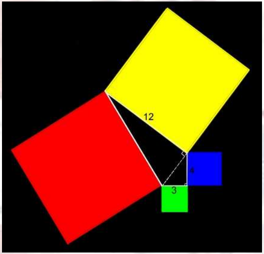


Splicing two familiar Pythagorean triples together we get:

$$3^2 + 4^2 + 12^2 = 13^2$$

There are other geometric intepretations of $x^2 + y^2 + z^2 = w^2$:

The area of the biggest square on a “staggered right angle” quadrilateral equals the sum of the areas of the smaller squares.



Then there are representation issues for positive integers:

Which numbers are sums of squares?

Actually, $n = x^2 + y^2 + z^2 + w^2$ has solutions for all $n \geq 0$, suspected at least as early at the 3rd century CE (Diophantus), and finally proved in 1770 (Joseph-Louis Lagrange).

Since 1789 (Aiden-Marie Legendre) we've known exactly when $n = x^2 + y^2 + z^2$ has solutions, and since 1625 (Albert Girard) when $n = x^2 + y^2$ does.

We also know how many representations there are for given n.

Since we know that every positive number is a sum of 4 squares, it's natural to wonder how many cubes, fourth powers, fifth powers, etc, are needed to represent all positive numbers.

For cubes we know that 9 are needed (1912, Arthur Wieferich and Aubrey Kempner).

For fourth powers it's 19 (1986, Ramachandran Balasubramanian *et al*).

For fifth powers it's 37 (1964, Chen Jungrun).

For sixth powers it's 73 (1940, Subbayya Sivasankaranarayana Pillai).

In general?

> *Q11. How many k-th powers are needed to represent all positive whole numbers as sums of such powers?*

It's conjectured to be $2^k + \left\lfloor (\frac{3}{2})^k \right\rfloor - 2$.

There is some circumstantial evidence: this formula holds for all $k \leq 471600000$ (1990, Kubina and Wunderlich).

12. Powerplay (from XYZ to ABC)

The equation $x^2 + y^2 = z^2$ suggests a plethora of also lovely generalisations involving higher powers, such as

Does $x^n + y^n = z^n$ have interesting solutions if $n > 2$?

This was one of the most famous open problems in mathematics until it was finally nailed in the mid 1990s (Andrew Wiles and Richard Taylor). The answer is no! (Fermat's Last Theorem)

The cases $n = 3, 4, 5, 7$ are older: 1770 (Euler), 1670 (Fermat), circa 1825 (Legendre, Dirichlet), and 1839 (Gabriel Lamé).

Once $n = 4$ was done it was of course only necessary to establish it for all odd primes $n = p$. However, these (and other) early proofs of special cases seem ungeneralisable.

Sophie Germain developed an approach which might (but didn't) work for all n. It did work for an infinite number of exponents.

We could seek solutions of longer equations, starting with cubes:

$$x^3 + y^3 + z^3 = w^3.$$

"Plato's number" $3^3 + 4^3 + 5^3 = 6^3$ is a noteworthy example. Ramanujan gave a formula that generates an infinite number of solutions (but does it give them all?). How about higher powers?

In 1911 (R. Norrie) uncovered this, the smallest such example:

$$30^4 + 120^4 + 272^4 + 315^4 = 353^4.$$

And this came to light in 1934 (Sastry):

$$7^5 + 43^5 + 57^5 + 80^5 + 100^5 = 107^5.$$

The smallest such example (1967, Lander, Parkin, Selfridge) is:

$$19^5 + 43^5 + 46^5 + 47^5 + 67^5 = 72^5.$$

Can such an equation hold where the number of variables on the LHS is less than the common exponent?

Euler thought not (1769)—cubes being a special case of FLT—and nobody doubted his instincts for a long time.

But the great master was wrong. In 1966 the first of three known counterexamples for fifth powers was discovered (Leon Lander and Thomas Parkin):

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5.$$

In

1986 Noam Elkies found a counterexample for fourth powers:

$$2682440^4 + 15365639^4 + 18796760^4 = 20615673^4.$$

Using elliptic curves Elkies also found a formula giving infinitely many like it. In 1988 Roger Frye found the smallest example:

$$95800^4 + 217519^4 + 414560^4 = 422481^4.$$

An example for seventh powers (1999, Mark Dodrill) is:

$$127^7 + 258^7 + 266^7 + 413^7 + 430^7 + 439^7 + 525^7 = 568^7.$$

A corresponding example for eighth powers (2000, Scott Chase) is:

$$90^8 + 223^8 + 478^8 + 524^8 + 748^8 + 1088^8 + 1190^8 + 1324^8 = 1409^8.$$

What about such an example for sixth powers?

Or ninth or larger powers?

None are known.

Can we mix powers in 3-term equations? Consider $x^s + y^t = z^u$ with $s \leq t$.

The case $t = 1$ is boring.

The case $x^0 + y^3 = z^2$ is more interesting—ignoring the fact that x can be anything—having a solution as easy as $(1, 2, 3)$.

In fact (2002, Mihăilescu) there are no other nontrivial solutions to $x^0 + y^t = z^u$ for $1 < t < u$ (Catalan's conjecture).

Does $x^s + y^t = z^u$ have solutions for $1 < s < t < u$? How about $x^3 + y^4 = z^5$ in particular? There are "trivial" solutions as well as less obvious ones that can be constructed using clever elementary methods. See *Superbrain* book (Diarmuid Early & Des MacHale).

Or, adding another term, how about $x^4 + y^5 + z^6 = w^7$?

A Beal Conjecture

The equations $2^n + 2^n = 2^{n+1}$ (any n), $3^3 + 6^3 = 3^5$, and $7^3 + 7^4 = 14^3$, are seeds for 3 infinite families of "similar" solutions to the equation $A^x + B^y = C^z$. What do they have in common?

A, B, C share common factors in each case.

So we can ask:

> *Q12 Does $A^x + B^y = C^z$ have solutions for A, B, C sharing no common factors, when x, y, z, all exceed 2?*

Texan billionaire Andy Beal conjectures not (1993), and is offering \$1 million for a proof or disproof.

The Beal conjecture trivially implies FLT.

The ABC conjecture

Can anything intelligent be said about the exponent 1 case?!

The equation $a + b = c$...

> *Q13 Does $a + b = c$ have solutions for a, b, c sharing no common factors, where the product d of the distinct prime factors of abc is not much smaller than c?*

We think not: so says the ABC conjecture (1985).

It can be shown that ABC implies Beal which implies FLT.

The fact that ABC implies FLT is almost trivial.

Why ABC implies FLT. (So what implies ABC?!)

> *Assume that if $a+b=c$ has solutions for a, b, c sharing no common factors, then the product d of the distinct prime factors of abc satisfies $c < d^2$*

Now suppose $x^n + y^n = z^n$ for some $n > 2$, where x, y, z share no common factors. Then taking $a = x^2$, $b = y^2$, $c = z^2$ we have the product d of the distinct prime factors of abc satisfies both

$$d \leq xyz < z^3 \text{ and } z^n < d^2$$

Hence $z^n < z^6$ and so $n < 6$. The $n = 3, 4, 5$ cases of FLT have long been known.

Can the ABC conjecture be generalized?

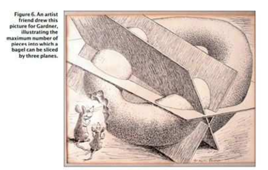
But of course! The n Conjecture (1994) ...

Score card

Problem	Easy	Tricky	Hard	Unknown	Unknowable
Fruit	X				
Doughnuts		X			
Bagels		X			
Cakes	X				
Coins					X
Box Cutting	X				
Bricks				X	
Series				X	
Ducks			X		
Primes				X	
SOS	X	X	X	X	
Powerplay	X	X	X	X	

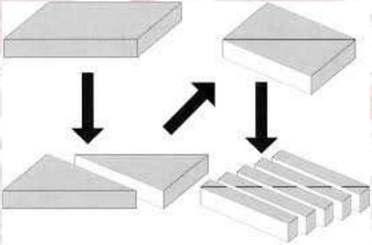
Have we won too many victories? (Say that out loud.)

So many questions! Here are three "divisive" answers:

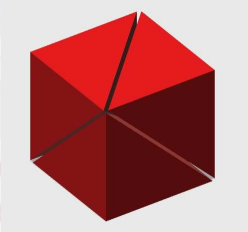


Q2. 13 doughnut pieces with 3 planar cuts:

Q4. Rectangular cake shared among 5 people



Q6. Rectangular box trisection into pyramids



From A to Ω

Math has an extensive track record of 1, 2, 3, ... generalisations:

1. The trip from solving linears & quadratics to solving cubics & quartics and trying (but failing, for good reason) to solve quintics occupies key chapters in its history. Likewise angle trisection.

2. Extending quadratic reciprocity ("does $x^2 = a$ have solutions mod p?") to cubic, quartic, etc, led to Artin's reciprocity law attack on Hilbert's Ninth Problem (cf. global class field theory).

3. In 2D, moving from deg 1 lines to deg 2 conic sections, to deg 3 elliptic curves and deg 4 lemniscates and cassini ovals and beyond. Which in turn (in 2D, 3D, etc) leads to algebraic geometry, real algebraic geometry and non-so-linear algebra (*aka* Gröbner bases).

4. Linear programming extends to quadratic programming, etc.

Those *One, Two, Many* journeys were difficult, technical, and took hundreds of years for some of the brightest minds to master.

We hope today's leisurely explorations have been more relaxing.