CONTINUED FRACTIONS WITH MULTIPLE LIMITS

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Abstract. For integers $m \geq 2$, we study divergent continued fractions whose numerators and denominators in each of the m arithmetic progressions modulo m converge. Special cases give, among other things, an infinite sequence of divergence theorems, the first of which is the classical Stern-Stolz theorem.

We give a theorem on a class of Poincaré type recurrences which shows that they tend to limits when the limits are taken in residue classes and the roots of their characteristic polynomials are distinct roots of unity.

We also generalize a curious q-continued fraction of Ramanujan's with three limits to a continued fraction with k distinct limit points, $k \geq 2$. The k limits are evaluated in terms of ratios of certain q series.

Finally, we show how to use Daniel Bernoulli's continued fraction in an elementary way to create analytic continued fractions with m limit points, for any positive integer $m \geq 2$.

1. Introduction

When one studies an infinite process, and it is found not to tend to a definite limit, an initial instinct is for one to discard the process as unsuitable. However, there are cases in which the divergence occurs in such a controlled way that the process still retains utility. Summability provides one example. Another way in which a divergent process can be useful is if it tends to a finite number of definite limits for "nice" subsequences of its approximants. This occurs in a natural way in the context of continued fractions, recurrence sequences and infinite products of matrices. Here we will make an intensive study of this behavior when the subsequences are arithmetic progressions of residue classes modulo m.

We begin by reviewing notation for continued fractions. The symbol K is used for continued fractions in the same way that $\sum$ and $\prod$ are used for

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series and products, respectively. Thus,

$$\underset{n=1}{\overset{N}{K}} \frac{a_n}{b_n} := \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ldots + \cfrac{a_N}{b_N}}}} = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots + \frac{a_N}{b_N}.$$

Write P_N/Q_N for the above finite continued fraction written as a rational function of the variables $a_1, ..., a_N, b_1, ..., b_N$. P_N is the *N-th canonical numerator*, Q_N is the *N-th canonical denominator* and the ratio P_N/Q_N is the *N-th approximant.* Let $\hat{\mathbb{C}}$ denote the extended complex plane.

By $K_{n=1}^{\infty} a_n/b_n$ we mean the limit of the sequence $\{P_n/Q_n\}$ as n tends to infinity, if the limit exists.

Two of the most interesting examples of continued fractions with more than one limit are due to Rogers-Ramanujan [10, 12], and Ramanujan [11], respectively:

$$1 + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \frac{q^4}{1} \cdots, \tag{1.1}$$

and

$$\frac{-1}{1+q} + \frac{-1}{1+q^2} + \frac{-1}{1+q^3} + \cdots. \tag{1.2}$$

Now, it is known that (1.1) converges for $|q| < 1$, while for $|q| > 1$, it tends to two different limits, depending on whether one considers the sequence of even or odd approximants. For (1.2), the behavior is even more interesting: it diverges for $|q| < 1$, but its sequence of approximants converge to different values depending on the residue class modulo 3 from which the approximants are chosen. We show that (1.1) and (1.2) are part of the same phenomenon– a phenomenon that we thoroughly explore here. We present a unified theory showing that this behavior is typical of a larger class of continued fractions which have multiple limits. We show that there is nothing special about two or three limits, and that continued fractions with $m \geq 2$ limits arise just as naturally.

In the course of our investigations it became necessary to investigate certain infinite products of matrices. These, in turn, led to theorems about the limiting behavior of Poincaré-type recurrences. We obtain a theorem similar to that of Perron on the limiting behavior of Poincaré-type recurrence sequences in the case where the eigenvalues are distinct roots of unity, but our theorem gives more information. We begin by describing this work.

1.1. **Poincaré-type recurrences.** Let the sequence $\{x_n\}_{n\geq 0}$ have the initial values $x_0, \ldots, x_{p-1}$ and subsequently defined by

$$x_{n+p} = \sum_{r=0}^{p-1} a_{n,r} x_{n+r}, \tag{1.3}$$

for $n \geq 0$. Suppose also that there are numbers $a_0, \ldots, a_{p-1}$ such that

$$\lim_{n\to\infty} a_{n,r} = a_r, \qquad 0 \leq r \leq p-1. \tag{1.4}$$

A recurrence of the form (1.3) satisfying the condition (1.4) is called a Poincaré-type recurrence, (1.4) being known as the Poincaré condition. Such recurrences were initially studied by Poincaré who proved that if the roots of the characteristic equation

$$t^p - a_{p-1}t^{p-1} - a_{p-2}t^{p-2} - \cdots - a_0 = 0 \tag{1.5}$$

have distinct norms, then the ratios of consecutive terms in the recurrence (for any set of initial conditions) tend to one of the roots. See [9]. Because the roots are also the eigenvalues of the associated companion matrix, they are also referred to as the eigenvalues of (1.3). This result was improved by O. Perron, who obtained a number of theorems about the limiting asymptotics of such recurrence sequences. Perron [8] made a significant advance in 1921 when he proved the following theorem which for the first time treated cases of eigenvalues which repeat or are of equal norm.

Theorem 1. *Let the sequence $\{x_n\}_{n\geq 0}$ be defined by initial values x_0, $\ldots$, x_{p-1} and by (1.3) for $n \geq 0$. Suppose also that there are numbers $a_0, \ldots, a_{p-1}$ satisfying (1.4). Let $q_1, q_2, \ldots q_\sigma$ be the distinct moduli of the roots of the characteristic equation (1.5) and let l_λ be the number of roots whose modulus is q_λ, multiple roots counted according to multiplicity, so that*

$$l_1 + l_2 + \ldots l_\sigma = p.$$

Then, provided $a_{n,0}$ be different from zero for $n \geq 0$, the difference equation (1.3) *has a fundamental system of solutions, which fall into σ classes, such that, for the solutions of the λ-th class and their linear combinations,*

$$\limsup_{n\to\infty} \sqrt[n]{|x_n|} = q_\lambda.$$

The number of solutions of the λ-th class is l_λ.

Thus when all of the characteristic roots have norm 1, this theorem gives that

$$\limsup_{n\to\infty} \sqrt[n]{|x_n|} = 1.$$

Another related paper is [6] where the authors study products of matrices and give a sufficient condition for their boundedness. This is then used to study "equimodular" limit periodic continued fractions, which are limit periodic continued fractions in which the characteristic roots of the associated 2×2 matrices are all equal in modulus. The matrix theorem in [6] can also be used to obtain results about the boundedness of recurrence sequences. We study a more specialized situation here and obtain far more detailed information as a consequence.

Our focus is on the case where the characteristic roots are distinct roots of unity. Under a condition stronger than (1.4) we will show that all non-trivial solutions of such recurrences approach a finite number of limits in a precisely controlled way. Specifically, our theorem is:

Theorem 2. *Let the sequence* $\{x_n\}_{n\geq 0}$ *be defined by initial values* x_0*,* $\ldots$*,* x_{p-1} *and by (1.3) for* $n \geq 0$*. Suppose also that there are numbers* $a_0, \ldots, a_{p-1}$ *such that*

$$\sum_{n=0}^{\infty} |a_r - a_{n,r}| < \infty, \qquad 0 \leq r \leq p-1.$$

Suppose the solutions of (1.5) are distinct roots of unity, say α_1*,* $\ldots$*,* α_p*. Let* m *be the least positive integer such that, for all* $j \in \{0, 1, \ldots, p-1\}$*,* $\alpha_j^m = 1$*. Then, for* $0 \leq j \leq m-1$*, the subsequence* $\{x_{mn+j}\}_{n=0}^{\infty}$ *converges. Set* $l_j = \lim_{n\to\infty} x_{nm+j}$*, for integers* $j \geq 0$*. Then the (periodic) sequence* $\{l_j\}$ *satisfies the recurrence relation*

$$l_{n+p} = \sum_{r=0}^{p-1} a_r l_{n+r}, \tag{1.6}$$

and thus there exist constants $c_1, \cdots, c_p$ *such that*

$$l_n = \sum_{i=1}^{p} c_i \alpha_i^n. \tag{1.7}$$

Remark. In [2] the authors study a recurrence which is of Poincaré type and has 6 limits. In section 4 we obtain this result of a special case of one of our corollaries.

Theorem 2 follows easily from Proposition 1 in section 2 which proves the convergence of infinite products of matrices when the limits are taken in arithmetic progressions. Proposition 1 is also our key to giving a unifying theory of certain classes of continued fractions with multiple limits.

1.2. **Continued fractions with multiple limits.** Our general theorem on continued fractions with multiple limits is the following which includes the multiple limit convergence behavior of both (1.1) and (1.2) as special cases.

Theorem 3. *Let* $\{p_n\}_{n\geq 1}$*,* $\{q_n\}_{n\geq 1}$ *be complex sequences satisfying*

$$\sum_{n=1}^{\infty} |p_n| < \infty, \qquad \sum_{n=1}^{\infty} |q_n| < \infty.$$

Let ω_1 and ω_2 be distinct roots of unity and let m be the least positive integer such that $\omega_1^m = \omega_2^m = 1$. Define

$$G := \frac{-\omega_1\omega_2 + q_1}{\omega_1+\omega_2+p_1} \, + \, \frac{-\omega_1\omega_2 + q_2}{\omega_1+\omega_2+p_2} \, + \, \frac{-\omega_1\omega_2 + q_3}{\omega_1+\omega_2+p_3} \, + \cdots.$$

Let $\{P_n/Q_n\}_{n=1}^{\infty}$ denote the sequence of approximants of G. If $q_n \neq \omega_1\omega_2$ for any $n \geq 1$, then G does not converge. However, the sequences of numerators and denominators in each of the m arithmetic progressions modulo m do converge. More precisely, there exist complex numbers $A_0, \ldots, A_{m-1}$ and $B_0, \ldots, B_{m-1}$ such that, for $0 \leq i < m$,

$$\lim_{k\to\infty} P_{m\,k+i} = A_i, \qquad \lim_{k\to\infty} Q_{m\,k+i} = B_i. \tag{1.8}$$

Extend the sequences $\{A_i\}$ and $\{B_i\}$ over all integers by making them periodic modulo m so that (1.8) continues to hold. Then for integers i,

$$A_i = \left(\frac{A_1 - \omega_2 A_0}{\omega_1 - \omega_2}\right)\omega_1^i + \left(\frac{\omega_1 A_0 - A_1}{\omega_1 - \omega_2}\right)\omega_2^i, \tag{1.9}$$

and

$$B_i = \left(\frac{B_1 - \omega_2 B_0}{\omega_1 - \omega_2}\right)\omega_1^i + \left(\frac{\omega_1 B_0 - B_1}{\omega_1 - \omega_2}\right)\omega_2^i. \tag{1.10}$$

Moreover,

$$A_iB_j - A_jB_i = -(\omega_1\omega_2)^{j+1}\frac{\omega_1^{i-j} - \omega_2^{i-j}}{\omega_1 - \omega_2}\prod_{n=1}^{\infty}\left(1 - \frac{q_n}{\omega_1\omega_2}\right). \tag{1.11}$$

Put $\omega_1 := \exp(2\pi i a/m)$, $\omega_2 := \exp(2\pi i b/m)$, $0 \leq a < b < m$, and $r := m/\gcd(b-a, m)$. Then G has r distinct limits in $\hat{\mathbb{C}}$ which are given by A_j/B_j, $1 \leq j \leq r$. Finally, for $k \geq 0$ and $1 \leq j \leq r$,

$$\frac{A_{j+kr}}{B_{j+kr}} = \frac{A_j}{B_j}.$$

To see how the behavior of (1.1) for $|q| > 1$ follows from this, observe that by the standard equivalence transformation for continued fractions, (1.1) is equal to

$$1 + \frac{1}{1/q} \, + \, \frac{1}{1/q} \, + \, \frac{1}{1/q^2} \, + \, \frac{1}{1/q^2} \cdots \, + \, \frac{1}{1/q^n} \, + \, \frac{1}{1/q^n} \cdots.$$

We can now apply Theorem 3 with $\omega_1 = -1$, $\omega_2 = 1$ (so $m = 2$), $q_n = 0$ and $p_{2n-1} = p_{2n} = 1/q^n$ to get that the continued fraction does not converge, but that the sequence of approximants in each of the arithmetic progressions modulo 2 do converge.

The behavior of (1.2) is similarly a special case. Put $\omega_1 = \exp(2\pi i/6)$, $\omega_2 = \exp(-2\pi i/6)$ (so that $\omega_1 + \omega_2 = \omega_1\omega_2 = 1$), $g_n = 0$ and $f_n = q^n$. Theorem 3 then gives that (1.2) has three limits for $|q| < 1$.

We refer to the number r in the theorem as the *rank* of the continued fraction. A remarkable consequence of this theorem is that because of (1.9) and (1.10), to compute all the limits of a continued fraction of rank r, one only needs to know the first two limits of the numerator and denominator convergents. In fact, taking $i = 0$ and $j = 1$ in (1.11), it can be seen that one only needs to know the value of three of the four limits $\{A_0, A_1, B_0, B_1\}$.

Another interesting consequence of Theorem 3 is that the fundamental Stern-Stolz divergence theorem [7] is an immediate corollary. In fact, the Stern-Stolz theorem will be found to be the beginning of an infinite sequence of similar theorems all of which are special cases of our theorem. See Corollaries 1-3 and Example 1. These consequences of Theorem 3 are explored after its proof.

1.3. **A generalization of the Ramanujan continued fraction with three limits.** In a recent paper [2], the authors proved a claim made by Ramanujan in his lost notebook ([11], p.45) about (1.2). To describe Ramanujan's claim, we first need some notation. Throughout take $q \in \mathbb{C}$ with $|q| < 1$. The following standard notation for q-products will also be employed:

$$(a)_0 := (a;q)_0 := 1, \quad (a)_n := (a;q)_n := \prod_{k=0}^{n-1}(1 - a\,q^k), \quad \text{if } n \geq 1,$$

and

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1 - a\,q^k), \qquad |q| < 1.$$

Set $\omega = e^{2\pi i/3}$. Ramanujan's claim was that, for $|q| < 1$,

(1.12)
$$\lim_{n\to\infty}\left(\frac{1}{1} - \frac{1}{1+q} - \frac{1}{1+q^2} - \cdots - \frac{1}{1+q^n+a}\right) = -\omega^2\left(\frac{\Omega - \omega^{n+1}}{\Omega - \omega^{n-1}}\right)\cdot\frac{(q^2;q^3)_\infty}{(q;q^3)_\infty},$$

where

$$\Omega := \frac{1-a\omega^2}{1-a\omega}\frac{(\omega^2 q, q)_\infty}{(\omega q, q)_\infty}.$$

Ramanujan's notation is confusing, but what his claim means is that the limit exists as $n \to \infty$ in each of the three congruence classes modulo 3, and that the limit is given by the expression on the right side of (1.12). Also, the appearance of the variable a in this formula is a bit of a red herring; from elementary properties of continued fractions, one can derive the result for general a from the $a = 0$ case.

Here we examine a direct generalization of Ramanujan's continued fraction which has k limits, for an arbitrary positive integer $k \geq 2$, and evaluate these limits in terms of ratios of certain unusual q-series. Let m be any arbitrary integer greater than 2, let ω be a primitive m-th root of unity and, for ease of notation, let $\bar{\omega} = 1/\omega$. Define

$$G(q) := \frac{1}{1} - \frac{1}{\omega+\bar{\omega}+q} - \frac{1}{\omega+\bar{\omega}+q^2} - \frac{1}{\omega+\bar{\omega}+q^3} + \cdots.$$

For $|q|<1$ and a, $x \neq 0$, define

$$P(a,x,q) := \sum_{j=0}^{\infty} \frac{q^{j(j+1)/2} a^j x^j}{(q;q)_j (x^2 q;q)_j}. \tag{1.13}$$

We prove the following theorem.

Theorem 4. *Let ω be a primitive m-th root of unity and let $\bar{\omega} = 1/\omega$. Let $1 \leq i \leq m$. Then*

$$\lim_{k\to\infty} \frac{1}{\omega+\bar{\omega}+q} - \frac{1}{\omega+\bar{\omega}+q^2} - \cdots \frac{1}{\omega+\bar{\omega}+q^{mk+i}} \tag{1.14}$$
$$= \frac{\omega^{1-i} P(q,\omega,q) - \omega^{i-1} P(q,w^{-1},q)}{\omega^{-i} P(1,\omega,q) - \omega^{i} P(1,w^{-1},q)}.$$

Moreover, the continued fraction has rank m when m is odd, and rank $m/2$ when m is even.

We state the result for the first tail of $G(q)$, rather than $G(q)$ itself, for aesthetic reasons.

We point out that this form of Theorem 4 is initially due to Ismail and Stanton [4]. Prior to becoming aware of their work, we had proved Theorem 4 in a form that involved the infinite series $F(\omega,i,j,q)$ of Lemma 6. Their result motivated us to prove Lemma 6, thereby deriving compact expressions for these series and thus arriving at the result of Theorem 4 by another route.

1.4. **Analytic continued fractions with multiple limits, via Daniel Bernoulli's Theorem.** In [2], the authors also describe a general class of analytic continued fractions with three limit points. Our final results are to give an alternative derivation, using Daniel Bernoulli's continued fraction, of this class of analytic continued fractions, and to generalize this class, again using Bernoulli's continued fraction, by showing how to construct analytic continued fractions with m limit points, for an arbitrary positive integer $m \geq 2$. Recall that Bernoulli's result is a simple formula for a continued fraction whose sequence of approximants agrees exactly with any prescribed sequence (See Proposition 2). If the original sequence is of a simple kind, then Bernoulli's continued fraction offers no advantage over the original sequence and so is in a certain sense "trivial". (It is just an obscure way of writing down a simple sequence.) Our results in Theorems 3 and 4 are deeper and do not arise from a simple sequence. Using Bernoulli's continued fraction we prove the following theorem which gives a general class of continued fractions having m limit points. The sequence equal to the n'th approximant of this continued fraction is constructed from taking the union of m sequences. This gives rise to the m limits. We have included this result to put into perspective the special case of Bernoulli's formula that is given in [2].

Theorem 5. *Let $G(z)$ be analytic in the closed unit disc and suppose*

$$|G(z)| < 1/2, \qquad |z| < 1. \tag{1.15}$$

Let m be a positive integer, $m \geq 2$. Define

$$\gamma_n = \begin{cases} 1-m, & n \equiv 0 (mod\ m), \\ 1, & otherwise. \end{cases} \tag{1.16}$$

Then, for $|z| < 1$, the continued fraction

$$\frac{G(z^2)-G(z)+1}{1} - \frac{G(z^3)-G(z^2)+1}{G(z^3)-G(z)+2} + K_{n=3}^{\infty} \frac{-(G(z^{n+1})-G(z^n)+\gamma_n)(G(z^{n-1})-G(z^{n-2})+\gamma_{n-2})}{G(z^{n+1})-G(z^{n-1})+\gamma_n+\gamma_{n-1}}. \tag{1.17}$$

has exactly the m limits $G(0)-G(z)+i$, $0 \leq i \leq m-1$.

2. A Result on Infinite Matrix Products

The convergence results of this paper follow from the following proposition.

Proposition 1. *Let $p \geq 2$ be an integer and let M be a $p \times p$ matrix that is diagonalizable and whose eigenvalues are roots of unity. Let I denote the $p \times p$ identity matrix and let m be the least positive integer such that*

$$M^m = I.$$

For a $p \times p$ matrix G, let

$$||G||_{\infty} = \max_{1 \leq i,j \leq p} |G^{(i,j)}|,$$

where $G^{(i,j)}$ denotes the element of G in row i and column j. Suppose $\{D_n\}_{n=1}^{\infty}$ is a sequence of matrices such that

$$\sum_{n=1}^{\infty} ||D_n - M||_{\infty} < \infty.$$

Then

$$F := \lim_{k\to\infty} \prod_{n=1}^{km} D_n$$

exists. Here the matrix product means either $D_1 D_2 \ldots$ or $\ldots D_2 D_1$. Further, for each j, $0 \leq j \leq m-1$,

$$\lim_{k\to\infty} \prod_{n=1}^{km+j} D_n = M^j F \text{ or } F M^j,$$

depending on whether the products are taken to the left or right.

We prove the proposition for the products $D_1 D_2 \ldots$ only, since the other case follows by taking the transpose. We need two preliminary lemmas.

Lemma 1. *For $n \geq 0$, define*

$$U_n = \prod_{j=1}^{m} D_{mn+j}.$$

Then there exists a sequence $\{\epsilon_n\}$ with $\sum_{n=0}^{\infty} \epsilon_n < \infty$ and an absolute constant A such that

$$||U_n - I||_\infty \leq \epsilon_n A.$$

Proof. Let $\epsilon_n = \max_{1 \leq j \leq m} ||D_{mn+j} - M||_\infty$. Define E_{mn+j} by

$$D_{mn+j} = M + \epsilon_n E_{mn+j}.$$

(If $\epsilon_n = 0$, define E_{mn+j} to be the $p \times p$ zero matrix). Note that the entries in each matrix E_{mn+j} are bounded in absolute value by 1. Let the matrix R_n be as defined below.

$$\begin{aligned} U_n = \prod_{j=1}^{m} D_{mn+j} &= \prod_{j=1}^{m} (M + \epsilon_n E_{mn+j}) \\ &:= M^m + \epsilon_n R_n = I + \epsilon_n R_n. \end{aligned}$$

The elements of all the matrices R_n for $n \geq 0$ are absolutely bounded (independent of n) since R_n is formed from a sum of at most 2^m products of matrices, where each product contains m matrices and the entries in each matrix are bounded by $\max\{||M||_\infty, \epsilon_n\}$. Let $A = \sup\{||R_n||_\infty\}$. Then

$$||U_n - I||_\infty = \epsilon_n ||R_n||_\infty \leq \epsilon_n A.$$

□

Lemma 2. *With the notation of the previous lemma, define*

$$F_r = \prod_{n=0}^{r} U_n.$$

Then $\lim_{r \to \infty} F_r$ *exists.*

Proof. Let A be as defined in the previous lemma.
Claim 1:

$$||F_r||_\infty \leq \prod_{j=0}^{r} (1 + p\epsilon_j A).$$

Proof of claim. For $r = 0$, $F_0 = U_0$ and

$$|F_0^{(i,j)}| = |U_0^{(i,j)}| \leq 1 + \epsilon_0 A \leq 1 + p\epsilon_0 A.$$

Assume the claim is true for $r = 0, 1, \ldots s$.

$$\begin{aligned} |F_{s+1}^{(i,j)}| = \left| \sum_{k=1}^{p} F_s^{(i,k)} U_{s+1}^{(k,j)} \right| &\leq \sum_{k=1}^{p} \left| F_s^{(i,k)} \right| \left| U_{s+1}^{(k,j)} \right| \leq \prod_{j=0}^{s} (1 + p\epsilon_j A) \sum_{k=1}^{p} \left| U_{s+1}^{(k,j)} \right| \\ &\leq \prod_{j=0}^{s} (1 + p\epsilon_j A)(1 + p\epsilon_{s+1} A) = \prod_{j=0}^{s+1} (1 + p\epsilon_j A). \end{aligned}$$

In particular, note that, for each $r \geq 0$ and each index (i, j),

$$|F_r^{(i,j)}| \leq \prod_{j=0}^{\infty} (1 + p\epsilon_j A) := C.$$

Note that the infinite product converges, since $\sum_{n=0}^{\infty} \epsilon_n \leq \sum_{n=0}^{\infty} ||D_n - M|| < \infty$.

Claim 2: For each index (i, j), the sequence $\{F_r^{(i,j)}\}$ is Cauchy, and hence convergent.

Proof of claim. By definition,

$$F_{r+1} - F_r = F_r U_{r+1} - F_r = F_r(U_{r+1} - I).$$

Hence,

$$\begin{aligned} |F_{r+1}^{(i,j)} - F_r^{(i,j)}| &= \left| \sum_{k=1}^{p} F_r^{(i,k)} (U_{r+1} - I)^{(k,j)} \right| \\ &\leq \sum_{k=1}^{p} |F_r^{(i,k)}| \left| (U_{r+1} - I)^{(k,j)} \right| \leq pCA\epsilon_{r+1}, \end{aligned}$$

where C is as defined immediately above. This is sufficient to show the sequence is Cauchy, since $\sum_{n=0}^{\infty} \epsilon_n < 0$. Define the matrix F by

$$F^{(i,j)} := \lim_{r \to \infty} F_r^{(i,j)}.$$

□

Proof of Proposition 1. This now follows easily from the above lemma, since

$$\begin{aligned} \lim_{k \to \infty} \prod_{n=1}^{km+j} D_n &= \lim_{k \to \infty} \prod_{n=1}^{km} D_n \prod_{n=km+1}^{km+j} D_n \\ &= \lim_{k \to \infty} F_{k-1} \prod_{n=km+1}^{km+j} D_n = FM^j. \end{aligned}$$

□

3. Continued Fractions With Multiple Limits

Proposition 1 allows us to construct non-trivial divergent continued fractions whose sequences of approximants in each of the arithmetic progressions modulo m converge. We now prove Theorem 3.

Proof of Theorem 3. Let

$$M = \begin{pmatrix} \omega_1 + \omega_2 & 1 \\ -\omega_1 \omega_2 & 0 \end{pmatrix}.$$

It follows easily from the identity

$$\begin{pmatrix} 1 & 1 \\ -\omega_2 & -\omega_1 \end{pmatrix} \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -\omega_2 & -\omega_1 \end{pmatrix}^{-1} = \begin{pmatrix} \omega_1+\omega_2 & 1 \\ -\omega_1\omega_2 & 0 \end{pmatrix}$$

that

$$M^j = \begin{pmatrix} \dfrac{{\omega_1}^{1+j} - {\omega_2}^{1+j}}{\omega_1 - \omega_2} & \dfrac{{\omega_1}^{j} - {\omega_2}^{j}}{\omega_1 - \omega_2} \\ -\dfrac{\omega_1\,\omega_2\,\left({\omega_1}^{j} - {\omega_2}^{j}\right)}{\omega_1 - \omega_2} & \dfrac{-{\omega_1}^{j}\,\omega_2 + \omega_1\,{\omega_2}^{j}}{\omega_1 - \omega_2} \end{pmatrix}, \tag{3.1}$$

and thus that

$$M^m = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad M^j \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad 1 \leq j < m.$$

For $n \geq 1$, define

$$D_n := \begin{pmatrix} \omega_1+\omega_2+p_n & 1 \\ -\omega_1\omega_2+q_n & 0 \end{pmatrix}.$$

Then

$$\sum_{n\geq 1} ||D_n - M||_\infty < \infty.$$

Further,

$$||D_n - M||_\infty = \max\{|p_n|, |q_n|\}.$$

Thus the matrix M and the matrices D_n satisfy the conditions of Proposition 1. Let the matrices F_i and F have the same meaning as in the proof of Proposition 1.

By the correspondence between matrices and continued fractions,

$$\begin{pmatrix} P_{mn+i} & P_{mn+i-1} \\ Q_{mn+i} & Q_{mn+i-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \prod_{j=1}^{mn+i} D_j \tag{3.2}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} F_{n-1} \prod_{j=mn+1}^{mn+i} D_j.$$

Now let $n \to \infty$ to get that

$$\lim_{n\to\infty} \begin{pmatrix} P_{mn+i} & P_{mn+i-1} \\ Q_{mn+i} & Q_{mn+i-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} F\,M^i. \tag{3.3}$$

This proves (1.8).

Now let $A_i := \lim_{n\to\infty} P_{mn+i}$, and $B_i := \lim_{n\to\infty} Q_{mn+i}$. Notice by definition that the sequences $\{A_i\}$ and $\{B_i\}$ are periodic modulo m.

It easily follows from (3.3) that

$$\begin{pmatrix} A_i & A_{i-1} \\ B_i & B_{i-1} \end{pmatrix} = \begin{pmatrix} A_j & A_{j-1} \\ B_j & B_{j-1} \end{pmatrix} M^{i-j}.$$

(3.1) also gives that

$$A_i = A_j \frac{{\omega_1}^{1+i-j} - {\omega_2}^{1+i-j}}{\omega_1 - \omega_2} - A_{j-1} \frac{\omega_1\,\omega_2\,\left({\omega_1}^{i-j} - {\omega_2}^{i-j}\right)}{\omega_1 - \omega_2}, \tag{3.4}$$

and

$$B_i = B_j \frac{{\omega_1}^{1+i-j} - {\omega_2}^{1+i-j}}{\omega_1 - \omega_2} - B_{j-1} \frac{\omega_1\,\omega_2\,\left({\omega_1}^{i-j} - {\omega_2}^{i-j}\right)}{\omega_1 - \omega_2}. \tag{3.5}$$

Thus

$$A_i B_j - A_j B_i = \frac{\left(A_j\,B_{-1+j} - A_{-1+j}\,B_j\right)\,\omega_1\,\omega_2\,\left({\omega_1}^{i-j} - {\omega_2}^{i-j}\right)}{\omega_1 - \omega_2}.$$

(1.9) and (1.10) follow from (3.4) and (3.5) by setting $j = 1$. (1.11) follows after applying the determinant formula

$$\begin{aligned} A_j B_{j-1} - A_{j-1} B_j &= -\lim_{k\to\infty} \prod_{n=1}^{mk+j} (\omega_1\omega_2 - q_n) \\ &= -(\omega_1\omega_2)^j \prod_{n=1}^{\infty} \left(1 - \frac{q_n}{\omega_1\omega_2}\right). \end{aligned}$$

Since $\sum_{j=1}^{\infty} |q_j|$ converges to a finite value, the infinite product on the right side converges.

For the continued fraction to converge, $A_i B_{i-1} - A_{i-1} B_i = 0$ is required. However, (1.11) shows that this is not the case.

Also from (1.11) we have that $A_i B_j - A_j B_i = 0$, and thus that $A_i/B_i = A_j/B_j$ in the extended complex plane, if and only if $\omega_1^{i-j} = \omega_2^{i-j}$. This happens if and only if

$$(i-j)a \equiv (i-j)b \mod m.$$

It follows easily from this that

$$\frac{A_j}{B_j}, \qquad 1 \le j \le \frac{m}{\gcd(b-a, m)} =: r,$$

are distinct and that

$$\frac{A_{j+kr}}{B_{j+kr}} = \frac{A_j}{B_j}, \qquad 1 \le j \le r, \qquad k \ge 0.$$

□

It is easy to derive general divergence results from this theorem, including the classical Stern-Stolz Theorem [7]. In fact, Stern-Stolz can be seen as the beginning of an infinite family of divergence theorems. We first derive the Stern-Stolz theorem as a corollary, generalize it, then give a corollary describing the infinite family. Last, we list the first few examples in the infinite family.

Corollary 1. *(Stern-Stolz) Let the sequence $\{b_n\}$ satisfy $\sum |b_n| < \infty$. Then*

$$b_0 + K_{n=1}^{\infty} \frac{1}{b_n}$$

diverges. In fact, for $p = 0, 1$,

$$\lim_{n\to\infty} P_{2n+p} = A_p \neq \infty, \qquad \lim_{n\to\infty} Q_{2n+p} = B_p \neq \infty,$$

and

$$A_1 B_0 - A_0 B_1 = 1.$$

Proof. This follows immediately from Theorem 3, upon setting $\omega_1 = 1$, $\omega_2 = -1$ (so $m = 2$), $q_n = 0$ and $p_n = b_n$. □

Note that by taking $q_n = a_n$ we immediately obtain a generalization.

Corollary 2. *Let the sequences $\{a_n\}$ and $\{b_n\}$ satisfy $a_n \neq -1$ for $n \geq 1$, $\sum |a_n| < \infty$ and $\sum |b_n| < \infty$. Then*

$$b_0 + K_{n=1}^{\infty} \frac{1 + a_n}{b_n}$$

diverges. In fact, for $p = 0, 1$,

$$\lim_{n\to\infty} P_{2n+p} = A_p \neq \infty, \qquad \lim_{n\to\infty} Q_{2n+p} = B_p \neq \infty,$$

and

$$A_1 B_0 - A_0 B_1 = \prod_{n=1}^{\infty} (1 + a_n).$$

Proof. This follows immediately from Theorem 3, upon setting $\omega_1 = 1$, $\omega_2 = -1$ (so $m = 2$), $q_n = a_n$ and $p_n = b_n$. □

We have not been able to find Corollary 2 in the literature.

A natural infinite family of Stern-Stolz type theorems is described by the following corollary.

Corollary 3. *Let the sequences $\{a_n\}$ and $\{b_n\}$ satisfy $a_n \neq 1$ for $n \geq 1$, $\sum |a_n| < \infty$ and $\sum |b_n| < \infty$. Let $m \geq 3$ and let ω_1 be a primitive m-th root of unity. Then*

$$b_0 + K_{n=1}^{\infty} \frac{-1 + a_n}{\omega_1 + \omega_1^{-1} + b_n}$$

does not converge, but the numerator and denominator convergents in each of the m arithmetic progressions modulo m do converge. If m is even, then for $1 \leq p \leq m/2$,

$$\lim_{n\to\infty} P_{mn+p} = -\lim_{n\to\infty} P_{mn+p+m/2} = A_p \neq \infty,$$
$$\lim_{n\to\infty} Q_{mn+p} = -\lim_{n\to\infty} Q_{mn+p+m/2} = B_p \neq \infty.$$

If m is odd, then the continued fraction has rank m. If m is even, then the continued fraction has rank $m/2$. Further, for $2 \leq p \leq m'$, where $m' = m$ if m is odd and $m/2$ if m is even,

$$A_p B_{p-1} - A_{p-1} B_p = -\prod_{n=1}^{\infty} (1 - a_n).$$

Proof. In Theorem 3, let $\omega_2 = 1/\omega_1$. □

Some explicit examples are given below.

Example 1. *Let the sequences $\{a_n\}$ and $\{b_n\}$ satisfy $a_n \neq 1$ for $n \geq 1$, $\sum |a_n| < \infty$ and $\sum |b_n| < \infty$. Then each of the following continued fractions diverges:*

(i) The following continued fraction has rank three:

$$b_0 + K_{n=1}^{\infty} \frac{-1 + a_n}{1 + b_n}. \tag{3.6}$$

In fact, for $p = 1, 2, 3$,

$$\lim_{n\to\infty} P_{6n+p} = -\lim_{n\to\infty} P_{6n+p+3} = A_p \neq \infty,$$
$$\lim_{n\to\infty} Q_{6n+p} = -\lim_{n\to\infty} Q_{6n+p+3} = B_p \neq \infty.$$

(ii) The following continued fraction has rank four:

$$b_0 + K_{n=1}^{\infty} \frac{-1 + a_n}{\sqrt{2} + b_n}. \tag{3.7}$$

In fact, for $p = 1, 2, 3, 4$,

$$\lim_{n\to\infty} P_{8n+p} = -\lim_{n\to\infty} P_{8n+p+4} = A_p \neq \infty,$$
$$\lim_{n\to\infty} Q_{8n+p} = -\lim_{n\to\infty} Q_{8n+p+4} = B_p \neq \infty.$$

(iii) The following continued fraction has rank five:

$$b_0 + K_{n=1}^{\infty} \frac{-1 + a_n}{(1 - \sqrt{5})/2 + b_n}. \tag{3.8}$$

In fact, for $p = 1, 2, 3, 4, 5$,

$$\lim_{n\to\infty} P_{5n+p} = A_p \neq \infty, \qquad \lim_{n\to\infty} Q_{5n+p} = B_p \neq \infty.$$

(iv) The following continued fraction has rank six:

$$b_0 + K_{n=1}^{\infty} \frac{-1 + a_n}{\sqrt{3} + b_n}. \tag{3.9}$$

In fact, for $p = 1, 2, 3, 4, 5, 6$,

$$\lim_{n\to\infty} P_{12n+p} = -\lim_{n\to\infty} P_{12n+p+6} = A_p \neq \infty,$$
$$\lim_{n\to\infty} Q_{12n+p} = -\lim_{n\to\infty} Q_{12n+p+6} = B_p \neq \infty.$$

In each case we have, for p in the appropriate range, that

$$A_p B_{p-1} - A_{p-1} B_p = -\prod_{n=1}^{\infty}(1-a_n).$$

Proof. In Corollary 3, set
(i) $\omega_1 = \exp(2\pi i/6)$;
(ii) $\omega_1 = \exp(2\pi i/8)$;
(iii) $\omega_1 = \exp(2\pi i/5)$;
(iv) $\omega_1 = \exp(2\pi i/12)$. □

The cases $\omega_1 = \exp(2\pi i/m)$, $m = 3, 4, 10$ give continued fractions that are the same as those above after an equivalence transformation and renormalization of the sequences $\{a_n\}$ and$\{b_n\}$. Note that the continued fractions (3.7) and (3.9) are, after an equivalence transformation and renormalizing the sequences $\{a_n\}$ and $\{b_n\}$, of the forms

$$b_0 + K_{n=1}^{\infty}\frac{-2+a_n}{2+b_n}, \tag{3.10}$$

and

$$b_0 + K_{n=1}^{\infty}\frac{-3+a_n}{3+b_n}, \tag{3.11}$$

respectively. Because of the equivalence transformations employed, the convergents do not tend to limits in (3.10) or (3.11). Also, it should be mentioned that Theorem 3.3 of [2] is essentially the special case $a_n = 0$ of part (i) of our example. Nevertheless (3.10) and (3.11) have ranks 4 and 6 respectively.

Theorem 3 now makes it trivial to construct q-continued fractions with arbitrarily many limits.

Example 2. *Let $f(x)$, $g(x) \in \mathbb{Z}[q][x]$ be polynomials with zero constant term. Let ω_1, ω_2 be distinct roots of unity and suppose m is the least positive integer such that $\omega_1^m = \omega_2^m = 1$. Define*

$$G(q) := \frac{-\omega_1\omega_2 + g(q)}{\omega_1+\omega_2+f(q)} + \frac{-\omega_1\omega_2 + g(q^2)}{\omega_1+\omega_2+f(q^2)} + \frac{-\omega_1\omega_2 + g(q^3)}{\omega_1+\omega_2+f(q^3)} + \cdots.$$

Let $|q| < 1$. If $g(q^n) \neq \omega_1\omega_2$ for any $n \geq 1$, then $G(q)$ does not converge. However, the sequences of approximants of $G(q)$ in each of the m arithmetic progressions modulo m converge to values in $\hat{\mathbb{C}}$. The continued fraction has rank $m/\gcd(b-a, m)$, where a and b are as defined in Theorem 3.

From this example we can conclude that (1.1) and (1.2) are far from unique examples and many other q-continued fractions with multiple limits can be immediately written down. Thus, to Ramanujanize a bit, one can immediately see that the continued fractions

$$\underset{n\geq 1}{\overset{\infty}{K}}\frac{-1/2}{1+q^n} \qquad \text{and} \qquad \underset{n\geq 1}{\overset{\infty}{K}}\frac{-1/2+q^n}{1+q^n} \tag{3.12}$$

both have rank four, while the continued fractions

$$\underset{n\geq 1}{\overset{\infty}{K}} \frac{-1/3}{1+q^n} \qquad \text{and} \qquad \underset{n\geq 1}{\overset{\infty}{K}} \frac{-1/3+q^n}{1+q^n} \tag{3.13}$$

both have rank six. We do not dwell further on q-continued fractions here, but in section 5 we will study a direct generalization of (1.2).

4. Recurrence relations with characteristic equations whose roots are roots of unity

Theorem 2 follows easily from Proposition 1. We now prove Theorem 2.

Proof of Theorem 2. Define

$$M := \begin{pmatrix} a_{p-1} & a_{p-2} & \dots & a_1 & a_0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

By the correspondence between polynomials and companion matrices, the eigenvalues of M are $\alpha_1, \dots, \alpha_p$, so that M is diagonalizable and satisfies

$$M^m = I, \qquad M^j \neq I, \qquad 1 \leq j \leq m-1.$$

For $n \geq 1$, define

$$D_n := \begin{pmatrix} a_{n-1,p-1} & a_{n-1,p-2} & \dots & a_{n-1,1} & a_{n-1,0} \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

Thus the matrices M and D_n satisfy the conditions of Proposition 1. From the recurrence relation at (4.2) we get

$$\begin{pmatrix} x_{mn+i+p-1} \\ x_{mn+i+p-2} \\ \vdots \\ x_{mn+i} \end{pmatrix} = \prod_{j=1}^{mn+i} D_j \begin{pmatrix} x_{p-1} \\ x_{p-2} \\ \vdots \\ x_0 \end{pmatrix}.$$

Let F have the same meaning as in Proposition 1 and then

$$\lim_{n\to\infty} \begin{pmatrix} x_{mn+i+p-1} \\ x_{mn+i+p-2} \\ \vdots \\ x_{mn+i} \end{pmatrix} = F\,M^i \begin{pmatrix} x_{p-1} \\ x_{p-2} \\ \vdots \\ x_0 \end{pmatrix}. \tag{4.1}$$

(1.6) now follows immediately by letting $n \to \infty$ in (1.3). This completes the proof. $\square$

When a specific M is known, (4.1) can sometimes be used to obtain further relations between the different limits. This is illustrated in the following corollary.

Corollary 4. *Let u and v be complex numbers, $(u,v) \neq (0,0)$ and let $\{a_n\}_{n\geq 1}$, $\{b_n\}_{n\geq 1}$ be sequences of complex numbers such that*

$$\sum_{n=1}^{\infty} |a_n| < \infty, \qquad \sum_{n=1}^{\infty} |b_n| < \infty.$$

Let ω_1 and ω_2 be distinct roots of unity and let m be the least positive integer such that $\omega_1^m = \omega_2^m = 1$. Let the sequence $\{x_n\}_{n\geq 0}$ be defined by $x_0 = u$, $x_1 = v$ and, for $n \geq 2$ by

$$x_n = (\omega_1 + \omega_2 + a_{n-1})x_{n-1} - (\omega_1\omega_2 + b_n)x_{n-2}. \tag{4.2}$$

Then,
(i) For fixed integer j the sequence $\{x_{mn+j}\}_{n\geq 0}$ is convergent;
(ii) If we set $l_j := \lim_{n\to\infty} x_{mn+j}$, then for integer j,

$$l_{j+1} = (\omega_1 + \omega_2)\, l_j - \omega_1\omega_2 l_{j-1};$$

(iii) If m is even and ω_1 and ω_2 are primitive m-th roots of unity, then

$$l_{m/2+j} = -l_j, \qquad 0 \leq j \leq m/2 - 1;$$

(iv)For $j \in \{1, 2, \ldots m-2\}$, at most one of l_{j-1}, l_j and l_{j+1} is zero.

Proof. Define

$$M = \begin{pmatrix} \omega_1 + \omega_2 & -\omega_1\omega_2 \\ 1 & 0 \end{pmatrix},$$

and, for $n \geq 1$, set

$$D_n = \begin{pmatrix} \omega_1 + \omega_2 + a_n & -\omega_1\omega_2 - b_n \\ 1 & 0 \end{pmatrix},$$

Statement (i) follows from the $p = 2$ case of Theorem 2, since the equation

$$t^2 - (\omega_1 + \omega_2)t + \omega_1\omega_2 = 0$$

has roots ω_1 and ω_2. Statement (ii) follows immediately from Theorem 2. Statement (iii) follows from the fact that under the given conditions, $M^{m/2} = -I$ and (4.1) gives

$$\begin{pmatrix} l_{m/2+j+1} \\ l_{m/2+j} \end{pmatrix} = F\, M^{m/2+j} \begin{pmatrix} u \\ v \end{pmatrix} = -F\, M^{j} \begin{pmatrix} u \\ v \end{pmatrix} = - \begin{pmatrix} l_{j+1} \\ l_j \end{pmatrix}.$$

If any two of l_{j-1}, l_j and l_{j+1} were zero, (iii) would then give that the third would also be zero. Thus

$$\left(\begin{array}{c} l_{j+1} \\ \\ l_j \end{array}\right) = \left(\begin{array}{c} 0 \\ \\ 0 \end{array}\right) = F\, M^j \left(\begin{array}{c} u \\ \\ v \end{array}\right),$$

which is a contradiction, since $det\, F = det M = 1$ and $(u, v) \neq (0, 0)$. $\square$

Theorem 3.1 from [2] follows from the above corollary, upon setting $\omega_1 = \exp(2\pi i/6)$, $\omega_2 = \exp(-2\pi i/6)$ and $b_n = 0$ for $n \geq 1$.

5. A Generalization of the Ramanujan Continued Fraction

We now study a generalization of Ramanujan's continued fraction (1.2). As above, let m be any arbitrary integer greater than 2, let ω be a primitive m-th root of unity and, for ease of notation, let $\bar{\omega} = 1/\omega$. Define

$$G(q) := \frac{1}{1} - \frac{1}{\omega + \bar{\omega} + q} - \frac{1}{\omega + \bar{\omega} + q^2} - \frac{1}{\omega + \bar{\omega} + q^3} + \cdots. \tag{5.1}$$

We let $P_N(q)/Q_N(q)$ denote the N-th approximant of $G(q)$. From Theorem 3, the sequence of approximants in each of the m arithmetic progressions modulo m converges (set $g(x) := 0$ and $f(x) = x$ in this theorem). We proceed initially along the same path as that followed by the authors in [2]. We recall the q-binomial theorem [1], pp. 35–36.

Lemma 3. *If* $\begin{bmatrix} n \\ m \end{bmatrix}$ *denotes the Gaussian polynomial defined by*

$$\begin{bmatrix} n \\ m \end{bmatrix} := \begin{bmatrix} n \\ m \end{bmatrix}_q := \begin{cases} \dfrac{(q;q)_n}{(q;q)_m (q;q)_{n-m}}, & \text{if } 0 \leq m \leq n, \\ 0, & \text{otherwise}, \end{cases}$$

then

$$(z;q)_N = \sum_{j=0}^{N} \begin{bmatrix} N \\ j \end{bmatrix} (-1)^j z^j q^{j(j-1)/2}, \tag{5.2}$$

$$\frac{1}{(z;q)_N} = \sum_{j=0}^{\infty} \begin{bmatrix} N+j-1 \\ j \end{bmatrix} z^j.$$

Lemma 4.

$$P_N(q) = \sum_{\substack{j,r,s \geq 0 \\ r+j+s+1=N}} q^{j(j+1)/2} \omega^{r-s} \begin{bmatrix} j+r \\ j \end{bmatrix} \begin{bmatrix} j+s \\ j \end{bmatrix}.$$

Proof. For $N \geq 2$, the sequence $\{P_N(q)\}$ satisfies

$$P_N(q) = (\omega + \bar{\omega} + q^{N-1}) P_{N-1}(q) - P_{N-2}(q), \tag{5.3}$$

with $P_1(q) = 1$ and $P_0(q) = 0$. Define

$$F(t) := \sum_{N=1}^{\infty} P_N(q)t^N.$$

If the recurrence relation (5.3) is multiplied by t^N and summed for $N \geq 2$, we have

$$F(t) - t = (\omega + \bar{\omega})tF(t) + tF(tq) - t^2F(t).$$

Thus

$$F(t) = \frac{t}{(1-\omega t)(1-\bar{\omega}t)} + \frac{t}{(1-\omega t)(1-\bar{\omega}t)}F(tq).$$

Iterating this equation and noting that $F(0) = 0$, we have that

$$\begin{aligned} F(t) &= \sum_{j=0}^{\infty} \frac{t^{j+1}q^{j(j+1)/2}}{(\omega t;q)_{j+1}(\bar{\omega}t;q)_{j+1}} \\ &= \sum_{j,r,s=0}^{\infty} t^{j+r+s+1}q^{j(j+1)/2}\omega^{r-s} \begin{bmatrix} j+r \\ j \end{bmatrix} \begin{bmatrix} j+s \\ j \end{bmatrix}, \end{aligned}$$

where the last equation follows from the second equation of (5.2). The result follows upon comparing coefficients of t^N. □

Lemma 5.

$$\begin{aligned} Q_N(q) = \sum_{\substack{j,r,s\geq 0 \\ r+j+s+1=N}} q^{j(j+1)/2}\omega^{r-s} \begin{bmatrix} j+r \\ j \end{bmatrix} \begin{bmatrix} j+s \\ j \end{bmatrix} - \\ \sum_{\substack{j,r,s\geq 0 \\ r+j+s+2=N}} q^{j(j+3)/2}\omega^{r-s} \begin{bmatrix} j+r \\ j \end{bmatrix} \begin{bmatrix} j+s \\ j \end{bmatrix}. \end{aligned}$$

Proof. The proof is similar to that of Lemma 4. For $N \geq 2$, the sequence $\{Q_N(q)\}$ satisfies

$$Q_N(q) = (\omega + \bar{\omega} + q^{N-1})Q_{N-1}(q) - Q_{N-2}(q), \tag{5.4}$$

with $Q_1(q) = Q_0(q) = 1$. Define

$$G(t) := \sum_{N=1}^{\infty} Q_N(q)t^N.$$

If the recurrence relation (5.3) is multiplied by t^N and summed for $N \geq 2$, we have

$$G(t) - t = (\omega + \bar{\omega})tG(t) + tG(tq) - t^2G(t) - t^2.$$

Thus

$$G(t) = \frac{t(1-t)}{(1-\omega t)(1-\bar{\omega}t)} + \frac{t}{(1-\omega t)(1-\bar{\omega}t)}G(tq).$$

Iterating this equation and noting that $G(0)=0$, we have that

$$\begin{aligned}
G(t) &= \sum_{j=0}^{\infty} \frac{t^{j+1}(1-tq^j)q^{j(j+1)/2}}{(\omega t;q)_{j+1}(\bar{\omega}t;q)_{j+1}} \\
&= \sum_{j,r,s=0}^{\infty} t^{j+r+s+1}(1-tq^j)q^{j(j+1)/2}\omega^{r-s} \begin{bmatrix} j+r \\ j \end{bmatrix} \begin{bmatrix} j+s \\ j \end{bmatrix},
\end{aligned}$$

where the last equation follows from the second equation of (5.2). The result follows upon comparing coefficients of t^N. □

Lemma 6. *Let ω be a primitive m-th root of unity and let $|q|<1$.*
(i) For $j \geq 0$, $k \geq 0$ and $i \in \mathbb{Z}$ define

$$G_k(\omega,i,j,q) := \sum_{u=0}^{mk+i} \omega^u (q^{u+1};q)_j.$$

Then

$$G(\omega,i,j,q) := \lim_{k\to\infty} G_k(\omega,i,j,q) = \frac{(q;q)_j}{(\omega q;q)_j(1-\omega)} - \frac{\omega^{i+1}}{1-\omega}. \tag{5.5}$$

(ii)For $j \geq 0$, $k \geq 0$ and $i \in \mathbb{Z}$ define

$$F_k(\omega,i,j,q) := \sum_{u=0}^{\lfloor \frac{mk+i}{2} \rfloor'} (q^{u+1};q)_j(\alpha^{2u-i}+\alpha^{i-2u}),$$

where the summation $\sum_{u=0}^{\lfloor \frac{mk+i}{2} \rfloor'}$ means that if $mk+i$ is even, the final term in the sum is $(q^{(mk+i)/2+1};q)_j$, rather than $2(q^{(mk+i)/2+1};q)_j$.
Then

$$\begin{aligned}
F(\omega,i,j,q) &:= \lim_{k\to\infty} F_k(\omega,i,j,q) \\
&= \frac{(q;q)_j}{\omega^{-1}-\omega}\left(\frac{\omega^{-i-1}}{(q\omega^2;q)_j} - \frac{\omega^{i+1}}{(q/\omega^2;q)_j}\right).
\end{aligned} \tag{5.6}$$

Proof. Clearly we may assume $0 \leq i \leq m-1$. The proof in each case is by induction on j.

(i) Both sides of (5.5) are easily seen to be true for $j=0$, since

$$\lim_{k\to\infty} \sum_{u=0}^{mk+i} \omega^u = \sum_{u=0}^{i} \omega^u.$$

Suppose $j \geq 1$ and that (5.5) is true for each i, with j replaced by $j-1$.

$$G(\omega,i,j,q) = \lim_{k\to\infty} \sum_{u=0}^{mk+i} \omega^u (q^{u+1};q)_j$$

$$= \lim_{k\to\infty} \sum_{u=0}^{mk+i} \omega^u (1-q^{u+1})(q^{u+2};q)_{j-1}$$

$$= \lim_{k\to\infty} \sum_{u=1}^{mk+i+1} \omega^{u-1}(1-q^u)(q^{u+1};q)_{j-1}$$

$$= \omega^{u-1}\left(\lim_{k\to\infty} \sum_{u=1}^{mk+i+1} \omega^u (q^{u+1};q)_{j-1} - \lim_{k\to\infty} \sum_{u=1}^{mk+i+1} (q^{u+1};q)_{j-1}(\omega q)^u\right)$$

$$= \omega^{u-1}\left(\lim_{k\to\infty} \sum_{u=0}^{mk+i+1} \omega^u (q^{u+1};q)_{j-1} - \lim_{k\to\infty} \sum_{u=0}^{mk+i+1} (q^{u+1};q)_{j-1}(\omega q)^u\right)$$

$$= \omega^{u-1}\left(G(\omega,i+1,j-1,q) - (q;q)_{j-1}\lim_{k\to\infty} \sum_{u=0}^{mk+i+1} \frac{(q;q)_{u+j-1}}{(q;q)_u(q;q)_{j-1}}(\omega q)^u\right)$$

$$= \omega^{-1}\left(\frac{(q;q)_{j-1}}{(\omega q;q)_{j-1}(1-\omega)} - \frac{\omega^{i+2}}{1-\omega} - \frac{(q;q)_{j-1}}{(\omega q;q)_j}\right).$$

The first equality follows from (5.5) and the second from (5.2). Some simple manipulations now give the result.

Remark: The proof of (i) is not necessary for the proof of our theorem and we give it for completeness only.

(ii) Equality for $j=0$ follows since

$$\lim_{k\to\infty} \sum_{u=0}^{\lfloor \frac{mk+i}{2}\rfloor'} (\omega^{2u-i}+\omega^{i-2u}) = \sum_{u=0}^{\lfloor \frac{i}{2}\rfloor'}(\omega^{2u-i}+\omega^{i-2u}) = \frac{\omega^{-i-1}-\omega^{i+1}}{\omega^{-1}-\omega}.$$

Now suppose $j \geq 1$ and (5.6) holds for each i and with j replaced by $j-1$.

$$F(\omega,i,j,q) = \lim_{k\to\infty} \sum_{u=0}^{\lfloor \frac{mk+i}{2}\rfloor'} (q^{u+1};q)_j(\omega^{2u-i}+\omega^{i-2u})$$

$$= \lim_{k\to\infty} \sum_{u=0}^{\lfloor \frac{mk+i}{2}\rfloor'} (1-q^{u+1})(q^{u+2};q)_{j-1}(\omega^{2u-i}+\omega^{i-2u})$$

$$= \lim_{k\to\infty} \sum_{u=0}^{\lfloor \frac{mk+i}{2}\rfloor'} (q^{u+2};q)_{j-1}(\omega^{2u-i}+\omega^{i-2u})$$

$$\quad - \lim_{k\to\infty} \sum_{u=0}^{\lfloor \frac{mk+i}{2}\rfloor'} (q^{u+2};q)_{j-1}(\omega^{2u-i}+\omega^{i-2u})q^{u+1}$$

$$= \lim_{k\to\infty} \sum_{u=1}^{\lfloor \frac{mk+i+2}{2}\rfloor'} (q^{u+1};q)_{j-1}(\omega^{2u-2-i}+\omega^{i+2-2u})$$

$$\quad - \lim_{k\to\infty} \sum_{u=1}^{\lfloor \frac{mk+i+2}{2}\rfloor'} (q^{u+1};q)_{j-1}(\omega^{2u-2-i}+\omega^{i+2-2u})q^u$$

$$
\begin{aligned}
&= \lim_{k\to\infty} \sum_{u=0}^{\lfloor \frac{mk+i+2}{2}\rfloor'} (q^{u+1};q)_{j-1}(\omega^{2u-2-i}+\omega^{i+2-2u}) \\
&\qquad - \lim_{k\to\infty} \sum_{u=0}^{\lfloor \frac{mk+i+2}{2}\rfloor'} (q^{u+1};q)_{j-1}(\omega^{2u-2-i}+\omega^{i+2-2u})q^u \\
&= F(\omega, i+2, j-1, q) \\
&\qquad - (q;q)_{j-1} \lim_{k\to\infty} \sum_{u=0}^{\lfloor \frac{mk+i+2}{2}\rfloor'} \frac{(q;q)_{u+j-1}}{(q;q)_u(q;q)_{j-1}}(\omega^{2u-2-i}+\omega^{i+2-2u})q^u \\
&= \frac{(q;q)_{j-1}}{\omega^{-1}-\omega}\left(\frac{\omega^{-i-3}}{(q\omega^2;q)_{j-1}} - \frac{\omega^{i+3}}{(q/\omega^2;q)_{j-1}}\right) \\
&\qquad\qquad - (q;q)_{j-1}\left(\frac{\omega^{-i-2}}{(q\omega^2;q)_j} + \frac{\omega^{i+2}}{(q/\omega^2;q)_j}\right).
\end{aligned}
$$

The first equality follows from (5.6) and the second follows from (5.2). A little algebraic manipulation now gives the result. $\square$

We note for later use that $|(q^r;q)_j - 1| \le 2^j|q|^r$ and hence, after subtracting the zero sum $\sum_{s=0}^{m-1}\omega^s$ from $F_{k+1}(\omega,i,j,q) - F_k(\omega,i,j,q)$, we get that

$$|F_{k+1}(\omega,i,j,q) - F_k(\omega,i,j,q)| \le m\,2^j|q|^{(mk+i)/2}. \tag{5.7}$$

Since $\lim_{k\to\infty} F_k(\omega,i,j,q) = F(\omega,i,j,q)$, it now follows that

$$|F(\omega,i,j,q) - F_k(\omega,i,j,q)| \le \frac{m\,2^j|q|^{(mk+i)/2}}{1-|q|^{m/2}}. \tag{5.8}$$

It is clear from the definition that $|F_0(\omega,i,j,q)| \le (|i|+2)2^j$, so that letting $k=0$ in (5.8) gives

$$|F(\omega,i,j,q)| \le 2^j\left(|i|+2+\frac{m|q|^{i/2}}{1-|q|^{m/2}}\right). \tag{5.9}$$

Lemma 7. *Let $1 \le i \le m$. With the notation of Lemma 6,*

$$\lim_{k\to\infty} P_{mk+i}(q) = \sum_{j=0}^{\infty} \frac{q^{j(j+1)/2}}{(q;\,q)_j^2} F(\omega, i-j-1, j, q). \tag{5.10}$$

$$
\begin{aligned}
\lim_{k\to\infty} Q_{mk+i}(q) &= \sum_{j=0}^{\infty} \frac{q^{j(j+1)/2}}{(q;\,q)_j^2} F(\omega, i-j-1, j, q) \\
&\qquad - \sum_{j=0}^{\infty} \frac{q^{j(j+3)/2}}{(q;\,q)_j^2} F(\omega, i-j-2, j, q).
\end{aligned}
\tag{5.11}
$$

Proof. From Lemma 4,

$$
\begin{aligned}
P_{mk+i}(q) &= \sum_{\substack{j,r,s\geq 0\\ r+j+s+1=mk+i}} q^{j(j+1)/2}\omega^{r-s}\begin{bmatrix} j+r\\ j\end{bmatrix}\begin{bmatrix} j+s\\ j\end{bmatrix}\\
&= \sum_{\substack{j,r,s\geq 0\\ r+j+s+1=mk+i}} \frac{q^{j(j+1)/2}}{(q;q)_j^2}\omega^{r-s}(q^{r+1};q)_j(q^{s+1};q)_j\\
&= \sum_{j=0}^{mk+i-1}\frac{q^{j(j+1)/2}}{(q;q)_j^2}\sum_{r=0}^{mk+i-j-1}\omega^{2r-(mk+i-j-1)}(q^{r+1};q)_j(q^{mk+i-j-r};q)_j\\
&= \sum_{j=0}^{mk+i-1}\frac{q^{j(j+1)/2}}{(q;q)_j^2}H_k(\omega,i,j,q),
\end{aligned}
$$

where

$$
\begin{aligned}
H_k(\omega,i,j,q) &:= \sum_{r=0}^{mk+i-j-1}\omega^{2r-(mk+i-j-1)}(q^{r+1};q)_j(q^{mk+i-j-r};q)_j\\
&= \sum_{r=0}^{\lfloor\frac{mk+i-j-1}{2}\rfloor'}(q^{r+1};q)_j(q^{mk+i-j-r};q)_j(\omega^{2r-(mk+i-j-1)}+\omega^{mk+i-j-1-2r})\\
&= \sum_{r=0}^{\lfloor\frac{mk+i-j-1}{2}\rfloor'}(q^{r+1};q)_j(q^{mk+i-j-r};q)_j(\omega^{2r-(i-j-1)}+\omega^{i-j-1-2r}).
\end{aligned}
$$

Here the summation $\sum_{r=0}^{\lfloor\frac{mk+i-j-1}{2}\rfloor'}$ has a meaning similar to that in Lemma 6, in that if $mk+i-j-1$ is even, then the final term is $(q^{\frac{mk+i-j+1}{2}};q)_j^2$, rather than $2(q^{\frac{mk+i-j+1}{2}};q)_j^2$. The sequence $\{(q^{r+1};q)_j\}_{r=0}^{\infty}$ is bounded by 2^j and $|\omega^{2r-(i-j-1)}+\omega^{i-j-1-2r}|\leq 2$. Thus

(5.12)

$$
\begin{aligned}
|F_k(\omega,i-j-1,j,q)-H_k(\omega,i,j,q)| &\leq 2^{j+1}\sum_{r=0}^{\lfloor\frac{mk+i-j-1}{2}\rfloor}|1-(q^{mk+i-j-r};q)_j|\\
&= 2^{j+1}\sum_{r=\lceil\frac{mk+i-j+1}{2}\rceil}^{mk+i-j}|1-(q^r;q)_j|\leq 2^{2j+1}\sum_{r=\lceil\frac{mk+i-j+1}{2}\rceil}^{mk+i-j}|q|^r\\
&\leq 2^{2j+1}\frac{|q|^{\lceil\frac{mk+i-j+1}{2}\rceil}}{1-|q|}.
\end{aligned}
$$

After applying the triangle inequality, we have that

$$\begin{aligned}
&\left|P_{mk+i}-\sum_{j=0}^{\infty}\frac{q^{j(j+1)/2}}{(q;q)_j^2}F(\omega,i-j-1,j,q)\right|\\
&\le\left|\sum_{j=mk+i}^{\infty}\frac{q^{j(j+1)/2}}{(q;q)_j^2}F(\omega,i-j-1,j,q)\right|\\
&\qquad+\left|\sum_{j=0}^{mk+i-1}\frac{q^{j(j+1)/2}}{(q;q)_j^2}(H_k(\omega,i,j,q)-F(\omega,i-j-1,j,q))\right|\\
&\le\sum_{j=mk+i}^{\infty}\frac{|q|^{j(j+1)/2}}{|(q;q)_j|^2}|F(\omega,i-j-1,j,q)|\\
&\qquad+\sum_{j=0}^{mk+i-1}\frac{|q|^{j(j+1)/2}}{|(q;q)_j|^2}|H_k(\omega,i,j,q)-F_k(\omega,i-j-1,j,q)|\\
&\qquad+\sum_{j=0}^{mk+i-1}\frac{|q|^{j(j+1)/2}}{|(q;q)_j|^2}|F_k(\omega,i-j-1,j,q)-F(\omega,i-j-1,j,q)|.
\end{aligned}$$

Now apply (5.9) to the first sum, (5.12) to the second sum, and (5.8) to the third sum to obtain

$$\begin{aligned}
&\left|P_{mk+i}-\sum_{j=0}^{\infty}\frac{q^{j(j+1)/2}}{(q;q)_j^2}F(\omega,i-j-1,j,q)\right|\\
&\le\sum_{j=mk+i}^{\infty}\frac{|q|^{j(j+1)/2}}{|(q;q)_j|^2}2^j\left(|i-j-1|+2+\frac{m|q|^{(i-j-1)/2}}{1-|q|^{m/2}}\right)+\\
&\sum_{j=0}^{mk+i-1}\frac{|q|^{j(j+1)/2}}{|(q;q)_j|^2}2^{2j+1}\frac{|q|^{\lceil\frac{mk+i-j+1}{2}\rceil}}{1-|q|}+\sum_{j=0}^{mk+i-1}\frac{|q|^{j(j+1)/2}}{|(q;q)_j|^2}\frac{m2^j|q|^{\frac{mk+i-j-1}{2}}}{1-|q|^{m/2}}.
\end{aligned}$$

The first sum is the tail of a convergent series and thus tends to 0 as $k\to\infty$. The second sum is majorized by the convergent series

$$\frac{|q|^{(mk+i-1)/2}}{1-|q|}\sum_{j=0}^{\infty}\frac{|q|^{j^2/2}2^{2j+1}}{|(q;q)_j|^2},$$

and the third sum is majorized by the convergent series

$$\frac{m|q|^{(mk+i-1)/2}}{1-|q|^{m/2}}\sum_{j=0}^{\infty}\frac{|q|^{j^2/2}2^j}{|(q;q)_j|^2}.$$

Both sums clearly tend to 0 also as $k\to\infty$, thus proving (5.10). The proof of (5.11) is virtually identical and so is omitted. □

We now prove Theorem 4.

Proof of Theorem 4. Theorem 3 establishes the rank of the continued fraction. The rest of the theorem follows immediately from Lemma 6, Lemma 7 and the definition of $P(a, x, q)$ at (1.13). $\square$

6. Constructing Analytic Continued Fractions with n limits, using Daniel Bernoulli's Continued Fraction

In 1775, Daniel Bernoulli [3] proved the following result (see, for example, [5], pp. 11–12).

Proposition 2. *Let $\{K_0, K_1, K_2, \ldots\}$ be a sequence of complex numbers such that $K_i \neq K_{i-1}$, for $i = 1, 2, \ldots$. Then $\{K_0, K_1, K_2, \ldots\}$ is the sequence of approximants of the continued fraction*

$$K_0 + \frac{K_1 - K_0}{1} + \frac{K_1 - K_2}{K_2 - K_0} + \frac{(K_1 - K_0)(K_2 - K_3)}{K_3 - K_1} + \cdots + \frac{(K_{n-2} - K_{n-3})(K_{n-1} - K_n)}{K_n - K_{n-2}} + \cdots. \tag{6.1}$$

Trivially, if $\lim_{k\to\infty} K_{mk+i} = L_i$, for $0 \leq i \leq m-1$, where each L_i is different, one has a continued fraction where the approximants in each of the m arithmetic progressions modulo m tend to a different limit. One easy way to use Bernoulli's continued fraction to construct continued fractions with arbitrarily many limits is as follows. Let m be a positive integer, $m \geq 2$. Let $\{a_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$, $\{d_n\}_{n=1}^{\infty}$ and $\{e_n\}_{n=1}^{\infty}$ be convergent sequences with non-zero limits a, c, d and e respectively. Define

$$K_{(n-1)m+j} := \frac{d_n + j\, e_n}{a_n + j\, c_n}$$

for $n \geq 1$ and $0 \leq j \leq m-1$. Provided $a + j\, c \neq 0$, for $0 \leq j \leq m-1$ and no two consecutive terms in the sequence $\{K_i\}$ are equal, then the continued fraction in (6.1) has the sequence of approximants

$$\left\{\frac{d_1}{a_1}, \frac{d_1 + e_1}{a_1 + c_1}, \frac{d_1 + 2e_1}{a_1 + 2c_1}, \ldots, \frac{d_1 + (m-1)e_1}{a_1 + (m-1)c_1}, \ldots, \right.$$
$$\left. \frac{d_n}{a_n}, \frac{d_n + e_n}{a_n + c_n}, \frac{d_n + 2e_n}{a_n + 2c_n}, \ldots, \frac{d_n + (m-1)e_n}{a_n + (m-1)c_n}, \ldots, \right\}.$$

Thus this continued fraction has exactly the following m limits :

$$\left\{\frac{d}{a}, \frac{d+e}{a+c}, \frac{d+2e}{a+2c}, \ldots, \frac{d+(m-1)e}{a+(m-1)c}\right\}.$$

In [2], the authors defined a general class of analytic continued fractions with three limit points as follows. Let F and G be meromorphic functions defined on the unit disc, $U := \{z \in \mathbb{C} : |z| < 1\}$, are analytic at the origin, and satisfy the functional equation,

$$F(z) + G(z) + zF(z)G(z) = 1. \tag{6.2}$$

Further assume that $z^n, n \geq 1$, is not a pole of either F or G. The following theorem was proved in [2].

Theorem 6. *Let F and G be meromorphic functions defined on U, as given above, which are analytic at the origin and satisfy the condition* (6.2). *Then the continued fraction*

(6.3)
$$\frac{1}{1} - \frac{1}{1+zF(z)} - \frac{1}{1+zG(z)} - \frac{1}{F(z)+G(z^2)} - \frac{1}{1+z^2F(z^2)} - \frac{1}{1+z^2G(z^2)} - \frac{1}{F(z^2)+G(z^3)} - \cdots - \frac{1}{1+z^nF(z^n)} - \frac{1}{1+z^nG(z^n)} - \frac{1}{F(z^n)+G(z^{n+1})} - \cdots$$

has exactly three limit points, L_0, L_1 and L_2. Moreover,

$$L_0 = \frac{z}{2z - zG(z) - 1}, \tag{6.4}$$

$$L_1 = \frac{z + z\,G(0) - 1}{(z + z\,G(0) - 1)\,(1 - G(z)) + (z-1)\,G(0)}, \tag{6.5}$$

$$L_2 = \frac{1 - z\,G(0)}{(1 - z\,G(0))\,(1 - G(z)) + (z-1)\,(1 - G(0))}. \tag{6.6}$$

We give an alternative proof, based on Proposition 2.

Proof. If we substitute for F from (6.2) and simplify the continued fraction, we get that the continued fraction in (6.3) is equivalent to

(6.7)
$$\frac{1}{1} - \frac{1+zG_1}{1+z} - \frac{1}{1} - \frac{1}{1 - G_1 + (1+zG_1)G_2} - \frac{(1+zG_1)(1+z^2G_2)}{1+z^2} - \cdots - \frac{1}{1} - \frac{1}{1 - G_{n-1} + (1+z^{n-1}G_{n-1})G_n} - \frac{(1+z^{n-1}G_{n-1})(1+z^nG_n)}{1+z^n} \cdots.$$

Here we use the notation G_n for $G(z^n)$. Define

$$a_n := 1,$$
$$c_n := \frac{-2 + 3\,z - (-1 + 2\,z)\,G_1 - (1 - 2\,z + z\,G_1)\,G_n + z^n\,(-1 + G_1)\,(z + G_n)}{2\,(1 - z + (1 - 2\,z + z^n)\,G_n + G_1\,(-1 + z + (z - z^n)\,G_n))},$$
$$d_n := \frac{1 - z + (-z + z^n)\,G_n}{1 - z + (1 - 2\,z + z^n)\,G_n + G_1\,(-1 + z + (z - z^n)\,G_n)},$$
$$e_n := \frac{-1 + 2\,z - z^{1+n} + (z - z^n)\,G_n}{2\,(1 - z + (1 - 2\,z + z^n)\,G_n + G_1\,(-1 + z + (z - z^n)\,G_n))}.$$

In (6.1), set $K_0 = 0$, and for $n \geq 0$, define

$$K_{3n+1} := \frac{d_{n+1}}{a_{n+1}}, \quad K_{3n+2} := \frac{d_{n+1} + e_{n+1}}{a_{n+1} + c_{n+1}}, \quad K_{3n+3} := \frac{d_{n+1} + 2e_{n+1}}{a_{n+1} + 2c_{n+1}}. \tag{6.8}$$

Then the continued fraction in (6.1) simplifies to give (6.7). If we simplify the expressions on the right side of (6.8), we have that this continued fraction has exactly the sequence of approximants

$$\left\{ \frac{1 - z + (-z + z^n)\, G_n}{1 - z + (1 - 2\,z + z^n)\, G_n + G_1\,(-1 + z + (z - z^n)\, G_n)}, \frac{1 - z^{1+n} + (-z + z^n)\, G_n}{-(z\,(-1 + z^n)) + (1 - 2\,z + z^n)\, G_n + G_1\,(-1 + z^{1+n} + (z - z^n)\, G_n)}, -\frac{z\,(-1 + z^n)}{-1 + 2\,z - z^{1+n} + z\,(-1 + z^n)\, G_1} \right\}_{n=1}^{\infty}. \tag{6.9}$$

Finally, we let $n \to \infty$ to get the three limits, noting that $z^n \to 0$ and $G_n = G(z^n) \to G(0)$. □

It is not difficult to create analytic continued fractions with m limit points, where m is an integer, $m \geq 2$. For example, Theorem 5 provides a continued fraction which is less convoluted in its contrivance than the one in Theorem 6. It is clear that Bernoulli's continued fraction can be used to construct many similar examples.

Proof of Theorem 5. In (6.1), put, for $i \geq 0$,

$$K_i = G(z^{i+1}) - G(z) + i (\text{mod } m).$$

The sequence $\{K_i\}_{i=0}^{\infty}$ has exactly the m limits stated in the theorem. For $i \geq 1$,

$$K_i - K_{i-1} = \begin{cases} G(z^{i+1}) - G(z^i) - m + 1, & \text{for } i \equiv 0 (\text{mod } m), \\ \\ G(z^{i+1}) - G(z^i) + 1, & \text{otherwise.} \end{cases} \tag{6.10}$$

(1.15) gives that $K_i - K_{i-1} \neq 0$ for $i \geq 1$. The continued fraction (1.17) above is simply Bernoulli's continued fraction (6.1) for the stated sequence $\{K_i\}_{i=0}^{\infty}$. □

7. Concluding Remarks

It should be noted that the condition in Theorem 3 that ω_1 and ω_2 be distinct is necessary, since otherwise the matrix M cannot be diagonalized. Moreover substituting $\omega_1 = \omega_2 = 1$ (so $m = 1$) into the continued fraction G in Theorem 3 does not necessarily give that G has $m = 1$ limit. An example of how dropping the condition that ω_1 and ω_2 be distinct can lead to a false result is provided by the continued fraction

$$K_{n=1}^{\infty}\frac{-1-4/(4n^2-1)}{2}=2K_{n=1}^{\infty}\frac{-1/4-1/(4n^2-1)}{1}.$$

Our Theorem 3 without the condition that ω_1 and ω_2 be distinct would predict that the first continued fraction above, and hence the second, would have one limit ($m=1$) and hence converge, but the second continued fraction diverges generally ([7], page 158).

Our formulas for the m limits in Theorem 4 lack the simplicity of Ramanujan's for the continued fraction with three limit points. Can the function $P(a,x,q)$ at (1.13) be expressed in terms of infinite products? Do the quotients of series on the right side of (1.14) have expressions in terms of infinite products?

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