m-DISSECTIONS OF SOME INFINITE PRODUCTS AND RELATED IDENTITIES

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ABSTRACT. For integers t and m with $m \geq 5$ relatively prime to 6 such that $1 \leq t < m/2$ and $\gcd(t, m) = 1$, define

$$Q(t,m) := \frac{(q^{2t}, q^{m-2t}, q^m; q^m)_\infty}{(q^t, q^{m-t}; q^m)_\infty}.$$

The m-dissection of this quintuple product is given in terms of other quintuple products. Hirschhorn's 5-dissections of Ramanujan's function $R = R(q)$ and its reciprocal follow as special cases of this general result.

Define the sequence $\{c_n\}_{n\geq 0}$ by

$$\frac{Q(t,m)}{(q^m; q^m)_\infty} =: \sum_{n=0}^{\infty} c_n q^n.$$

It is proven that

$$n = n_0 := \frac{1}{6}(m-4)m(m+1) + (m-1)t$$

is the largest value of n for which $c_n = 0$ (a slight modification to the value of n_0 is needed if $t = (m-1)/2$). Further, it is shown that for $n > n_0$, the signs of the c_n are periodic with period m. In addition, a formula is given for a polynomial $f_{t,m}(q)$ (deriving from the m-dissection of $Q(t,m)$) from which a second polynomial $g_{t,m}(q)$ of degree $m-1$ is derived, such that the coefficient of q^d (either 1 or -1) in $g_{t,m}(q)$ indicates the sign of all the coefficients in the arithmetic progression c_{mk+d}, for $0 \leq d \leq m-1$ and $mk + d > n_0$.

By using methods like those used to prove the results above, other similar results are proved. We re-derive the result of Evans and Ramanathan giving the m-dissection of $(q;q)_\infty$ in terms of quintuple products for any integer $m \equiv \pm 1 \pmod 6$.

Other results include various Lambert series identities, such as the following. Define

$$A = A(q) := \frac{(q^2, q^5; q^7)_\infty}{(q, q^6; q^7)_\infty}, \quad B = B(q) := \frac{(q^3, q^4; q^7)_\infty}{(q^2, q^5; q^7)_\infty}.$$

Then

$$1 + 14\sum_{n=1}^{\infty} \frac{q^{3n}}{1-q^{7n}} - 14\sum_{n=1}^{\infty} \frac{q^{4n}}{1-q^{7n}} = \left(A^3 + \frac{3Bq}{A} - \frac{6q}{B}\right) \frac{f_7^2}{(q^3, q^4; q^7)_\infty},$$

where $f_i := (q^i; q^i)_\infty$.

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1. INTRODUCTION

In the present paper the quintuple product identity is used to derive m dissections of various infinite products, where m is a positive odd integer that is not a multiple of 3. These m-dissections are then used to derive various results concerning the non-vanishing of coefficients in the series expansion of certain infinite products, and also to show the m-periodicity in the pattern of the signs of these coefficients.

One family of results gives the m-dissection of Euler's product $(q;q)_\infty$ in terms of quintuple products for all positive integral $m \equiv \pm 1 \pmod 6$. This result was derived previously by Ramanathan [16, Theorem 1, page124] and Evans [7] (see also [3, Theorem 12.1, page 274]). However, we include a proof for completeness, and also because we use a slightly different formulation of the result to derive results about the periodicity of signs in the series expansion of a related infinite product. Specifically, by employing the idea that Hirschhorn used in [11], we show that if the sequence $\{c_n\}$ is defined by

$$\frac{(q;q)_\infty}{(q^m;q^m)_\infty} =: \sum_{n=0}^{\infty} c_n q^n,$$

then for $n > n_0(m)$ ($n_0(m)$ is explicitly computable) the signs of the c_n are periodic modulo m, and the pattern of $+/-$ signs can be stated exactly (although the c_n in roughly half of all arithmetic progressions modulo m are identically zero).

In [11] Hirschhorn gave the 5-dissection of the Ramanujan function $R(q)$ (see (3.6)) and its reciprocal (which are both essentially quintuple products) in terms of quintuple products (see Corollary 3.1 for the statement of these expansions). In the present paper this result is extended to all integers m relatively prime to 6. Specifically, if $m > 1$ is an integer that is relatively prime to 6 and t is an integer such that $1 \le t < m/2$ and $\gcd(t,m) = 1$, then the m-dissection of the quintuple product

$$Q(t,m) := \frac{(q^{2t}, q^{m-2t}, q^m; q^m)_\infty}{(q^t, q^{m-t}; q^m)_\infty} \tag{1.1}$$

is given in terms of other quintuple products.

Remark: The restriction to t satisfying $\gcd(t,m) = 1$, while necessary for the proof, still permits for the m-dissection of $Q(t,m)$ to be found in the case $\gcd(t,m) = d > 1$. To see this, observe that $Q(t/d, m/d)$ will satisfy the requirements of Theorem 3.1, so that its m/d-dissection may be found, and hence the m-dissection of $Q(t,m)$ may be determined by applying the dilation $q \to q^d$.

Further, if the sequence $\{c_n\}_{n\ge 0}$ is defined by

$$\frac{Q(t,m)}{(q^m;q^m)_\infty} =: \sum_{n=0}^{\infty} c_n q^n,$$

it is shown that

$$n = n_0 := \frac{1}{6}(m-4)m(m+1) + (m-1)t$$

is the largest value of n for which $c_n = 0$ (the value for n_0 needs to be slightly modified if $t = (m-1)/2$ - see Theorem 3.2). Further, it is shown that for $n > n_0$, the signs of the c_n are periodic with period m. In addition, a formula is given for a polynomial $f_{t,m}(q)$ (which is derived from the expansion of $Q(t,m)$ in terms of quintuple products) from which a second polynomial $g_{t,m}(q)$ of degree $m-1$ is derived, such that the coefficient of q^d (either 1 or -1) in $g_{t,m}(q)$ indicates the sign of all the coefficients in the arithmetic progression c_{mk+d}, for $0 \leq d \leq m-1$ and $mk + d > n_0$.

Remark: In [16], Ramanathan showed how to find, for each arithmetic progression $l \pmod{m}$, an integer $n_0(l)$ such that if $n = mk + l \geq n_0(l)$, then $c_n \neq 0$, and all integers in the set $\{c_{mk+l} : k \geq 0, mk + l \geq n_0(l)\}$ have the same sign. Thus Ramanathan's result for the periodicity of the signs of the coefficients is more refined than ours, but ours is easier to apply. Ramanathan did not provide the m dissection in terms of quintuple products that we give.

If there is more than one representation for an m-dissection of some infinite product (for example, there are three such m-dissection for $(q;q)_\infty^3$), then further identities may be derived by equating corresponding parts in two representations. A number of Lambert series identities follow in this way. One example of such an identity (which we believe to be new) is the following:

$$1 + 14\sum_{n=1}^{\infty} \frac{q^{3n}}{1-q^{7n}} - 14\sum_{n=1}^{\infty} \frac{q^{4n}}{1-q^{7n}} = \left(A^3 + \frac{3Bq}{A} - \frac{6q}{B}\right)\frac{f_7^2}{(q^3,q^4;q^7)_\infty},$$

where

$$A = A(q) := \frac{(q^2,q^5;q^7)_\infty}{(q,q^6;q^7)_\infty},\ B = B(q) := \frac{(q^3,q^4;q^7)_\infty}{(q^2,q^5;q^7)_\infty},\ f_i := (q^i;q^i)_\infty.$$

Another identity which follows similarly is

$$\sum_{k=0}^{\infty}(-1)^k\left((14k+1)q^{\frac{7k^2}{2}+\frac{k}{2}} + (14k+13)q^{\frac{7k^2}{2}+\frac{13k}{2}+3}\right) = f_7^3\left(A^3 - \frac{6q}{B} + \frac{3Bq}{A}\right).$$

We note that there are many other dissections of infinite products in the literature which do not follow from the methods in the present paper (see also the discussion preceding Theorem 3.1). Dou and Xiao [6], for example, have given the 5-dissections of

$$(-q,-q^4;q^5)_\infty(q,q^9;q^{10})_\infty^3 \qquad \text{and} \qquad (-q^2,-q^4;q^5)_\infty(q^3,q^7;q^{10})_\infty^3,$$

and used their results to reprove results of Hirschhorn [14] on vanishing coefficients.

Another difference with the present paper is that elsewhere the authors have often proved their results, as did Dou and Xiao in the paper mentioned just above [6], by appealing to properties of Ramanujan's general theta function $f(a,b) = (-a,-b,ab;ab)_\infty$ and the properties of particular cases of this function:

$$\phi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \tag{1.2}$$

$$\psi(q) := f(q,q^3) = \sum_{n=-\infty}^{\infty} q^{2n^2-n} = \sum_{n=0}^{\infty} q^{n(n+1)/2}, \tag{1.3}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}. \tag{1.4}$$

Yet another method of proving dissection results involves manipulation of bilateral series, as was employed by Hirschhorn [13] to derive the 2- and 4-dissections of Ramanujan's continued fraction and its reciprocal. This method was not employed either in the present paper.

Yet another method, again different from the methods in the present paper, is to use known dissections of simpler products to derive the dissections of more complicated products. This was the method used by Hirschhorn and Roselin [15] to obtain the 2-, 3-, 4- and 6-dissections of Ramanujans cubic continued fraction $RC(q)$, where

$$RC(q) := \frac{(q,q^5;q^6)_\infty}{(q^3,q^3;q^6)_\infty}. \tag{1.5}$$

1.1. **Notation and Preliminary Results.** Before discussing results, we recall some notation.

$$(a;q)_\infty := \prod_{n=0}^{\infty} 1 - aq^n$$
$$(a_1,\dots,a_j;q)_\infty := (a_1;q)_\infty \cdots (a_j;q)_\infty$$

For space-saving reasons, define

$$\langle a;q^k\rangle_\infty := (a,q^k/a,q^k;q^k)_\infty, \qquad f_i := (q^i;q^i)_\infty.$$

For ease of notation, we refer to any quantity of the form $\langle a;q^k\rangle_\infty$ as a *Jacobi triple product* or *triple product*.

Frequent use is made of the Jacobi triple product identity, so we recall two equivalent versions:

Theorem 1.1. *For $|q|<1$ and $z \neq 0$,*

$$\sum_{n=-\infty}^{\infty} (-z)^n q^{n^2} = \langle zq;q^2\rangle_\infty. \tag{1.6}$$

$$\sum_{n=-\infty}^{\infty} (-z)^n q^{n(n-1)/2} = \langle z; q \rangle_\infty. \tag{1.7}$$

Also central to the paper is the quintuple product identity (see [5] for an excellent survey article on the quintuple product identity and its many proofs):

$$\begin{aligned}\sum_{n=-\infty}^{\infty} q^{n(3n-1)/2} z^{3n}(1 - zq^n) &= \langle -qz^3; q^3 \rangle_\infty - z\langle -q^2 z^3; q^3 \rangle_\infty \\ &= \langle z; q \rangle_\infty (qz^2, q/z^2; q^2)_\infty = \frac{\langle z^2; q \rangle_\infty}{(-z, -q/z; q)_\infty}.\end{aligned} \tag{1.8}$$

Also for ease of notation we refer to a quantity expressed in any of the forms in (1.8) as a *quintuple product*. For later use, for integers t and m, with $m \geq 5$ and $1 \leq t < m/2$, $Q(t, m)$ will be as defined at (1.1). Note that $Q(t, m)$ is derived from the infinite product on the right side of (1.8) by replacing q with q^m and setting $z = -q^t$.

If a q-product is expressible as a sum/product of Jacobi triple products, it is easy to derive an m-dissection in terms of other Jacobi triple products. As is well-known, the following m-dissection arises from separating the bilateral series in Theorem 1.1 into m arithmetic progressions.

Corollary 1.1. *If $|q| < 1$, $z \neq 0$ and m is a positive integer, then*

$$\langle zq; q^2 \rangle_\infty = \sum_{r=0}^{m-1} (-z)^r q^{r^2} \left\langle (-1)^{m+1} z^m q^{m^2+2mr}; q^{2m^2} \right\rangle_\infty. \tag{1.9}$$

$$\langle z; q \rangle_\infty = \sum_{r=0}^{m-1} (-z)^r q^{(r^2-r)/2} \left\langle (-1)^{m+1} z^m q^{(m^2-m)/2+mr}; q^{m^2} \right\rangle_\infty. \tag{1.10}$$

However, m-dissections arising directly from these expansions often do not provide much information about, for example, the vanishing- and sign-periodicity of the coefficients in the series expansion of the original infinite product. While it is not hard to see that the m-dissection of the general quintuple product at (1.8) may easily be derived in terms of triple products, by applying (1.10) to the two triple products on the left side of (1.8), it is not possible to express the m-dissection of this general quintuple product in terms of other quintuple products (or at least not in a straightforward manner). However, somewhat surprisingly it is possible to do this for quite a large class of quintuple products derived from this general quintuple product, when q is replaced with q^m and z is specialized to be a power of q (see Theorem 3.1).

Identities of the following types, particularly the third (which combines the first two) are used often in what follows:

$$(q^{a-k}; q^k)_\infty = (1 - q^{a-k})(q^a; q^k)_\infty, \tag{1.11}$$

$$(q^{b+k}; q^k)_\infty = \frac{(q^b; q^k)_\infty}{1-q^b},$$
$$(q^{-b}, q^{b+k}; q^k)_\infty = -q^{-b}(q^b, q^{k-b}; q^k)_\infty.$$

2. The m-dissection of $(q;q)_\infty$ in terms of quintuple products

A special case of (1.10) follows upon making the "dilation" $q \to q^3$ and then making the substitution $z = q$ (noting that $(q, q^2, q^3; q^3)_\infty = (q;q)_\infty$):

$$(q;q)_\infty = \sum_{r=0}^{m-1} (-1)^r q^{(3r^2-r)/2} \left\langle (-1)^{m+1} q^{(3m^2-m)/2+3mr}; q^{3m^2} \right\rangle_\infty . \tag{2.1}$$

As remarked above, Evans [7] and Ramanathan [16] derived an alternative expression, in terms of quintuple products, for the m-dissection of $(q;q)_\infty$ in the case $m \geq 5$ is an odd integer relatively prime to 3. For completeness and because we require a slightly different formulation of this result to prove statements about the signs of coefficients, we also give a proof here. The method of proof is essentially that given by Berndt [3, Theorem 12.1, page 274], although our formulation is a little different (different summation variables and we have q where Berndt has $q^{1/m}$, for example).

The aim is to combine pairs of terms in the sum (2.1) using the quintuple product identity, so the next thing is to examine when

$$(3r^2 - r)/2 \equiv (3s^2 - s)/2 \pmod{m}, \tag{2.2}$$

when $s \neq r$ and $0 \leq r, s \leq m-1$ and $m > 3$ is relatively prime to 6. It turns out that the behaviour for integers of the form $6t-1$ is different from that for integers of the form $6t+1$. As illustrations, we consider the lists

$$\left\{ \frac{3r^2 - r}{2} \pmod{m} \right\}_{r=0}^{m-1}$$

for $m = 17$ and $m = 35$ (a prime and a composite of the form $6t-1$) and $m = 19$ and $m = 25$ (a prime and a composite of the form $6t+1$).

For 17, this list is

$$\{\underbrace{0, 1, 5, 12, 5, 1, 0}, \underbrace{2, 7, 15, 9, 6, 6, 9, 15, 7, 2}\}.$$

It can be seen that the list contains two palindromic sublists, the first being from $r = 0$ to $r = 6$ $(= (m+1)/3)$, and the second being from $r = 7$ $(= (m+1)/3 + 1)$ to $r = 16$ $(= m-1)$. Note that the first sublist has an unmatched term, 12, at $r = 3$ $(= (m+1)/6)$, while the second sublist does not. The same decomposition into two palindromic sublists occurs for composite integers of the form $6t-1$, the difference being that some integers occur more than twice, in contrast to the situation for primes. For $m = 35$, for example the list is

$$\{\underbrace{0,1,5,12,22,0,16,0,22,12,5,1,0}, \\ \underbrace{2,7,15,26,5,22,7,30,21,15,12,12,15,21,30,7,22,5,26,15,7,2}\}$$

However, this multiple occurrence of some integers does not affect the combination of terms via the quintuple product identity.

For 19, this list is

$$\{\underbrace{0,1,5,12,3,16,13,13,16,3,12,5,1,0},\underbrace{2,7,15,7,2}\}.$$

Here also it can be seen that the list contains two palindromic sublists, the first this time being from $r = 0$ to $r = 13$ $(= 2(m-1)/3+1)$, and the second being from $r = 14$ $(= 2(m-1)/3+2)$ to $r = 18$ $(= m-1)$. Note this time that the second sublist has an unmatched term, 15, at $r = 16$ $(= 5(m-1)/6+1)$, while the first sublist does not. For $m = 25$ the list is

$$\{\underbrace{0,1,5,12,22,10,1,20,17,17,20,1,10,22,12,5,1,0},\underbrace{2,7,15,1,15,7,2}\}$$

It will be seen that pattern of decomposition into palindromic sublists exhibited by 17 and 35 is replicated for all positive integers m of the form $m = 6t-1$, and the pattern exhibited by 19 and 25 is replicated for all positive integers m of the form $m = 6t+1$.

Theorem 2.1. *Let* $|q| < 1$ *and* $Q(t,m)$ *be as at* (1.1).

(i) If m *is a positive integer of the form* $6t+1$*, then*

$$(q;q)_\infty = (-1)^{(m-1)/6} q^{(m^2-1)/24} \left(q^{m^2};q^{m^2}\right)_\infty \\ + \sum_{r=0}^{(m-1)/3} (-1)^r q^{r(3r-1)/2} Q\left(m(m+6r-1)/6;m^2\right) \\ + \sum_{u=1}^{(m-7)/6} (-1)^u q^{u(3u+1)/2} Q\left(m(m-6u-1)/6;m^2\right). \tag{2.3}$$

(ii) If m *is a positive integer of the form* $6t-1$*, then*

$$(q;q)_\infty = (-1)^{(m+1)/6} q^{(m^2-1)/24} \left(q^{m^2};q^{m^2}\right)_\infty \\ + \sum_{r=0}^{(m-5)/6} (-1)^r q^{r(3r-1)/2} Q\left(m(m-6r+1)/6;m^2\right) \\ + \sum_{u=1}^{(m-2)/3} (-1)^u q^{u(3u+1)/2} Q\left(m(m+6u+1)/6;m^2\right). \tag{2.4}$$

Proof. We prove (ii) and omit the proof of (i) as it is similar. First consider when (2.2) holds. Since $(3r^2-r)/2-(3s^2-s)/2 = (r-s)(-1+3r+3s)/2$ then $(3r^2-r)/2 \equiv (3s^2-s)/2 \pmod m$ if $m|(-1+3r+3s)$. Further, given the restrictions on the size of r and s and the fact that m is a positive integer of the form $6t-1$, this implies that there are two possible cases. The first is that $-1+3r+3s=m$, or $s=(m+1)/3-r$, giving the matching pairs of values in the sum of terms in the first palindromic sublist, for $r=0,1,\dots,(m+1)/6-1$ (the unmatched central term, for $r=(m+1)/6$, also needs to be determined).

The second possibility is that $-1+3r+3s=4m$ and since $s\leq m-1$, then $r\geq (m+1)/3+1$. So writing $r=(m+1)/3+u$, then $s=m-u$, for $u=1,2,\dots(m+1)/3-1$, for the sum of terms in the second palindromic sublist.

The term in (2.1) corresponding to $s=(m+1)/3-r$ is (noting that $(-1)^{1-m}=(-1)^{1+m}=1, (-1)^{(1+m)/3-r}=(-1)^r$)

$$(-1)^r q^{(m-3r)(m-3r+1)/6}\left\langle q^{m(5m-6r+1)/2};q^{3m^2}\right\rangle_\infty.$$

Thus adding the terms for r and s in the first palindromic sublist in (2.1) together, one gets

$$\begin{aligned}
&(-1)^r q^{(3r^2-r)/2}\left\langle q^{(3m^2-m)/2+3mr};q^{3m^2}\right\rangle_\infty\\
&\qquad\qquad+(-1)^r q^{(m-3r)(m-3r+1)/6}\left\langle q^{m(5m-6r+1)/2};q^{3m^2}\right\rangle_\infty\\
&=(-1)^r q^{(3r^2-r)/2}\Bigg[\left\langle q^{(3m^2-m)/2+3mr};q^{3m^2}\right\rangle_\infty\\
&\qquad\qquad\qquad\qquad+q^{m(1+m-6r)/6}\left\langle q^{m(5m-6r+1)/2};q^{3m^2}\right\rangle_\infty\Bigg]\\
&\qquad\qquad=(-1)^r q^{(3r^2-r)/2}\frac{\left\langle q^{m(m-6r+1)/3};q^{m^2}\right\rangle_\infty}{\left(q^{m(m-6r+1)/6},q^{m(5m+6r-1)/6};q^{m^2}\right)_\infty},
\end{aligned}$$

where the last equality follows from the quintuple product identity (1.8) with q replaced with q^{m^2} and $z=-q^{m(1+m-6r)/6}$. This gives the first sum at (2.4) above.

The term in (2.1) corresponding to $s=m-u$ is

$$\begin{aligned}
(2.5)\quad &(-1)^{m-u}q^{(3m-3u-1)(m-u)/2}\left\langle q^{-m(3m-6u-1)/2};q^{3m^2}\right\rangle_\infty\\
&\qquad\qquad=(-1)^u q^{u(3u+1)/2}\left\langle q^{m(3m-6u-1)/2};q^{3m^2}\right\rangle_\infty.
\end{aligned}$$

Here the elementary identities of the type at (1.11) have been employed.

Thus adding the terms corresponding to $r=(m+1)/3+u$ and $s=m-u$ gives

$$\begin{aligned}
&(-1)^u q^{(m+3u)(m+3u+1)/6}\left\langle q^{m(m-6u-1)/2};q^{3m^2}\right\rangle_\infty\\
&\qquad\qquad+(-1)^u q^{u(3u+1)/2}\left\langle q^{m(3m-6u-1)/2};q^{3m^2}\right\rangle_\infty\\
&\qquad\qquad\qquad=(-1)^u q^{u(3u+1)/2}
\end{aligned}$$

$$\times \left[\left\langle q^{m(3m-6u-1)/2}; q^{3m^2} \right\rangle_\infty + q^{m(m+6u+1)/6} \left\langle q^{m(m-6u-1)/2}; q^{3m^2} \right\rangle_\infty \right]$$
$$= (-1)^u q^{u(3u+1)/2} \frac{\left\langle q^{m(m+6u+1)/3}; q^{m^2} \right\rangle_\infty}{\left(q^{m(m+6u+1)/6}, q^{m(5m-6u-1)/6}; q^{m^2}\right)_\infty},$$

where once again the last equality follows from the quintuple product identity (1.8) with q replaced with q^{m^2} and this time with $z = -q^{m(m+6u+1)/6}$. This gives the second sum at (2.4) above.

Finally, the isolated term in (2.4) follows from setting $r = (m+1)/6$ in the general term in (2.1), and simplifying.

The proof of (i) is similar, and the details are omitted. We remark however that since this time m has the form $m = 6t+1$ for some integer t, then $m|(-1+3r+3s)$ implies that either $-1+3r+3s = 2m$, or $-1+3r+3s = 5m$. $\square$

The well-known 5-, 7- and 11-dissections follow immediately from Theorem 2.1.

Corollary 2.1. *If $|q| < 1$, then*

(2.6)

$$(q;q)_\infty = (q^{25};q^{25})_\infty \left[\frac{(q^{10},q^{15};q^{25})_\infty}{(q^5,q^{20};q^{25})_\infty} - q - q^2 \frac{(q^5,q^{20};q^{25})_\infty}{(q^{10},q^{15};q^{25})_\infty} \right],$$
$$= (q^{49};q^{49})_\infty \left[\frac{(q^{14},q^{35};q^{49})_\infty}{(q^7,q^{42};q^{49})_\infty} - q \frac{(q^{21},q^{28};q^{49})_\infty}{(q^{14},q^{35};q^{49})_\infty} - q^2 + q^5 \frac{(q^7,q^{42};q^{49})_\infty}{(q^{21},q^{28};q^{49})_\infty} \right],$$
$$= (q^{121};q^{121})_\infty \left[\frac{(q^{44},q^{77};q^{121})_\infty}{(q^{22},q^{99};q^{121})_\infty} - q \frac{(q^{22},q^{99};q^{121})_\infty}{(q^{11},q^{110};q^{121})_\infty} \right.$$
$$\left. - q^2 \frac{(q^{55},q^{66};q^{121})_\infty}{(q^{33},q^{88};q^{121})_\infty} + q^5 + q^7 \frac{(q^{33},q^{88};q^{121})_\infty}{(q^{44},q^{77};q^{121})_\infty} - q^{15} \frac{(q^{11},q^{110};q^{121})_\infty}{(q^{55},q^{66};q^{121})_\infty} \right].$$

Proof. The 5- and 11-dissections follow from setting $m = 5$ and 11, respectively, in (2.4), and the 7-dissection follows from setting $m = 7$ in (2.3). $\square$

Remarks: (1) From the representation

$$\langle zq; q^2 \rangle_\infty = \sum_{n=-\infty}^{\infty} (-z)^n q^{n^2} = 1 + \sum_{n=1}^{\infty} (-1)^n q^{n^2} \left(z^n + z^{-n}\right) \tag{2.7}$$

it can be seen that the series expansion of any triple product derived from $\langle zq; q^2 \rangle_\infty$ by a dilation of the form $q \to q^m$ ($m > 1$ a positive integer) followed by a replacement of the form $z \to \pm q^t$ (t a positive integer) is lacunary. This means that the series expansion of any quintuple product $Q(t,m)$ is also lacunary, since any such $Q(t,m)$ is derived from a sum of two triple products. This applies to each of the quintuple products in the m-dissections in Theorems 2.1 and 3.1, and thus in particular to the quintuple products in the 5-, 7- and 11-dissections in Corollary 2.1, the quintuple products in Corollary 3.1, and elsewhere..

Remark: By applying (1.11) to the $m-r$-th term in (2.1), for $r=1,2,\dots\lfloor(m-1)/2\rfloor$, one also derives that

(2.8)

$$
\begin{aligned}
(q;q)_\infty &= \langle q^{m(-1+3m)/2};q^{3m^2}\rangle_\infty \\
&+\sum_{r=1}^{(m-1)/2}(-1)^r\Big[q^{r(-1+3r)/2}\langle q^{m(-1+3m+6r)/2};q^{3m^2}\rangle_\infty \\
&\qquad\qquad + q^{r(1+3r)/2}\langle q^{m(-1+3m-6r)/2};q^{3m^2}\rangle_\infty\Big], \qquad m \text{ odd},\\
&=\langle -q^{m(-1+3m)/2};q^{3m^2}\rangle_\infty + (-1)^{m/2}q^{m(3m-2)/8}\langle -q^{m/2};q^{3m^2}\rangle \\
&+\sum_{r=1}^{m/2-1}(-1)^r\Big[q^{r(-1+3r)/2}\langle -q^{m(-1+3m+6r)/2};q^{3m^2}\rangle_\infty \\
&\qquad\qquad + q^{r(1+3r)/2}\langle -q^{m(-1+3m-6r)/2};q^{3m^2}\rangle_\infty\Big], \qquad m \text{ even}.
\end{aligned}
$$

This expansion has the advantage that it is not limited to $m \equiv \pm 1 \pmod 6$, but has the disadvantage that it gives just an $m/2$-dissection when m is even.

We next prove a result about the periodicity of the signs of the coefficients in the series expansion of $(q;q)_\infty/(q^m;q^m)_\infty$, where $m \geq 5$ is relatively prime to 6. It is also shown that in those arithmetic progressions modulo m in which the coefficients in the series expansion of the above product are not all identically zero, that the coefficients are all eventually non-vanishing. These results follow easily from Theorem 2.1, but as far as we are aware, they have not been written down (possibly because Theorem 2.1 was previously stated in terms of $q^{1/m}$, rather than q, as in the present paper).

Theorem 2.2. *Let $m \geq 5$ be relatively prime to 6 and define the sequence c_n by*

$$
\frac{(q;q)_\infty}{(q^m;q^m)_\infty} =: \sum_{n=0}^{\infty} c_n q^n. \tag{2.9}
$$

In both the cases $m \equiv 1 \pmod 6$ and $m \equiv 5 \pmod 6$, define

$$
n_0 := \frac{1}{6}(m+1)(m+2). \tag{2.10}
$$

If $m \equiv 1 \pmod 6$, define

(2.11)

$$
f_m(q) = (-1)^{\frac{m-1}{6}} q^{\frac{1}{24}(m^2-1)} + \sum_{r=0}^{\frac{m-1}{3}} (-1)^r q^{\frac{1}{2}r(3r-1)} + \sum_{u=1}^{\frac{m-7}{6}} (-1)^u q^{\frac{1}{2}u(3u+1)}.
$$

If $m \equiv 5 \pmod 6$, *define*

(2.12)
$$f_m(q) = (-1)^{\frac{m+1}{6}} q^{\frac{1}{24}(m^2-1)} + \sum_{r=0}^{\frac{m-5}{6}} (-1)^r q^{\frac{1}{2}r(3r-1)} + \sum_{u=1}^{\frac{m-2}{3}} (-1)^u q^{\frac{1}{2}u(3u+1)}.$$

In either case, let

$$g_m(q) := \sum_{j=0}^{m-1} d_j q^j \tag{2.13}$$

be the polynomial derived from $f_m(q)$ *by reducing all the exponents modulo* m *(extended to have degree* $m-1$ *by introducing zero coefficients where necessary). Let* j *be an integer satisfying* $0 \le j \le m-1$.

(i) If $d_j = 0$, *then* $c_{mk+j} = 0$ *for all integers* $k \ge 0$.

(ii) $d_j \neq 0$, *then* $c_{mk+j} \neq 0$ *for all integers* $k \ge 0$ *such that* $mk + j > n_0$. *Moreover, for all such* k, *sign(*d_j*) = sign(*c_{mk+j}*).*

Proof. Observe that for each of the cases $m \equiv 1 \pmod 6$ and $m \equiv 5 \pmod 6$, that $f_m(q)$ is derived from the corresponding m-dissection in Theorem 2.1 by setting each $Q(\_,\_) = 1$.

Note also that since all of the non-zero coefficients in $f_m(q)$ have the form $(-1)^u$ for some integer u, then each $d_j \in \{-1, 0, 1\}$. (Recall that the exponents of the powers of q in front of the $Q(\_,\_)$ in the corresponding m-dissections in Theorem 2.1 are all distinct modulo m by construction, and thus all the powers of q in $f_m(q)$ are distinct modulo m also.)

Next we consider the case $m \equiv 1 \pmod 6$ and examine an arbitrary term in the m-dissection of the quotient $(q;q)_\infty/(q^m;q^m)_\infty$ derived from (2.3) (similar reason applies to the case $m \equiv 5 \pmod 6$ and (2.4), so the proof in this case is omitted). We consider a term coming from the first sum in (2.3):

$$\begin{aligned}(-1)^r q^{r(3r-1)/2} \frac{Q\left(a, m^2\right)}{(q^m;q^m)_\infty} \\ = (-1)^r q^{r(3r-1)/2} \frac{(q^{2a}, q^{m^2-2a}, q^{m^2}; q^{m^2})_\infty}{(q^a, q^{m^2-a}; q^{m^2})_\infty (q^m;q^m)_\infty}\end{aligned} \tag{2.14}$$

where for ease of notation we have written a for $m(m+6r-1)/6$. Since $(q^m;q^m)_\infty = (q^m, q^{2m}, \dots, q^{m^2-m}, q^{m^2}; q^{m^2})_\infty$, the product may be rewritten as

$$\frac{1}{(q^a, q^{m^2-a}; q^{m^2})_\infty \prod_{j=1}^{(*)m} (q^{jm}; q^{m^2})_\infty},$$

where the (*) indicates that $jm \neq 2a, m^2-2a$ or m^2. From this representation it may be seen that all coefficients in the series expansion (regarded as a series in powers of q^m) of the last infinite product are non-negative. In fact, in all but one exceptional case ($r = (m-1)/3$), all coefficients are strictly positive. This is the case since the factor $(q^m;q^{m^2})_\infty$ is not cancelled in the expansion of $(q^m;q^m)_\infty$, since neither $2a$ nor m^2-2a is equal to m, the reason being that $0 \leq r \leq (m-1)/3$, and therefore

$$\frac{1}{6}m(m-1) \leq a = \frac{m(m+6r-1)}{6} \leq \frac{1}{2}m(m-1). \tag{2.15}$$

It can thus be seen that in the series expansion of $(q;q)_\infty/(q^m;q^m)_\infty$, all terms in the sub-sequence $\{c_{mk+r(3r-1)/2}\}$ (apart from $r=(m-1)/3$) are non-zero for $k \geq 0$ and all have sign $(-1)^r$, and all terms vanish for any k for which $0 \leq mk + r(3r-1)/2 < r(3r-1)/2$.

It may be seen from (2.15) that the one exceptional case is $r=(m-1)/3$, at the upper end of the summation, and in this case $a=m(m-1)/2$ and hence $m^2-2a=m$. In this case the term on the right side of (2.14) becomes

$$q^{(m-1)(m-2)/6}\frac{(q^m,q^{m^2-m},q^{m^2};q^{m^2})_\infty}{(q^{m(m-1)/2},q^{m(m+1)/2};q^{m^2})_\infty(q^m;q^m)_\infty}.$$

Thus the factor $(q^m;q^{m^2})_\infty$ is cancelled from the denominator of this expression, the coefficient of $q^{(m-1)(m-2)/6+m}=q^{(m+1)(m+2)/6}=q^{n_0}$ (where n_0 is as defined at (2.10)) in the series expansion of this product is zero, and the coefficients of all higher powers of q in the same arithmetic progression module m are non-zero (since $(q^{2m},q^{3m};q^{m^2})_\infty$ is <u>not</u> cancelled from the denominator).

Two additional facts are needed to complete the proof in the case $m \equiv 1 \pmod 6$, both of which are easily checked. The first is that the exceptional situation just described does not occur with any term in the second sum at (2.3). Thus a situation similar to that described following (2.15) holds for all terms in this second sum.

The second fact is that the value of r in the exceptional case described above, namely, $r=(m-1)/3$, is the value that gives the largest of any of the powers of q of the form $q^{r(3r-1)/2}$ or $q^{u(3u+1)/2}$ found in the two sums at (2.3).

From what has just been shown, it can be seen that in those arithmetic progressions modulo m in which c_n is not identically zero for all n, that $c_n \neq 0$ if $n > n_0$.

The claims at (i) and (ii) above now follow from what has already been shown above.

As already mentioned above, the proofs of the statements for $m \equiv 5 \pmod 6$ are similar, and are omitted. □

We consider $f_{31}(q)$ as an example.

Example 1. *Let* $m=31$ *in* (2.11) *to get*

$$\begin{aligned} f_{31}(q) = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} \\ - q^{35} - q^{40} + q^{51} - q^{70} + q^{92} - q^{117} + q^{145}. \end{aligned} \tag{2.16}$$

Reduce the coefficients modulo 31 to get

$$g_{31}(q) = 1 - q - q^2 - q^4 + q^5 + q^7 - q^8 - q^9 - q^{12} \\ - q^{15} + q^{20} + q^{21} + q^{22} - q^{24} + q^{26} + q^{30}. \tag{2.17}$$

From the value $n_0 = (31+1)(31+2)/6 = 176$ *and the pattern of signs/zero coefficients in* $g_{31}(q)$*, it can be seen that if the sequence* $\{c_n\}$ *is defined by*

$$\frac{(q;q)_\infty}{(q^{31};q^{31})_\infty} =: \sum_{n=0}^{\infty} c_n q^n,$$

then the pattern of the signs of the c_n *in the arithmetic progression* $31k+d$*, for* $d = 0, 1, \ldots, 29, 30$*, respectively, and* $31k + d > 176$*, are given, respectively, by*

$$+, -, -, 0, -, +, 0, +, -, -, 0, 0, -, 0, 0, -, \\ 0, 0, 0, 0, +, +, +, 0, -, 0, +, 0, 0, 0, +.$$

Here a "0" in a particular position indicates that all coefficients in the arithmetic progression corresponding to that position are identically zero. Note that the statement above also means that if $n > 176$ *and* $n \equiv a \pmod{31}$*, where*

$$a \in \{0, 1, 2, 4, 5, 7, 8, 9, 12, 15, 20, 21, 22, 24, 26, 30\},$$

then $c_n \neq 0$.

3. m-DISSECTIONS OF QUINTUPLE PRODUCTS IN TERMS OF OTHER QUINTUPLE PRODUCTS

We next turn to the question of when the m-dissection of a quintuple product may be expressed in terms of other quintuple products. It is not difficult to see that this cannot be done (or at least not in a straightforward manner) for the general quintuple product on the right side of (1.8). However, it can be done in a large class of special cases, as demonstrated in the next theorem.

Before coming to this theorem, we note that it gives the m-dissection of $Q(t,m)$ only in the case where $m \geq 5$ and m is relatively prime to 6. Other methods may give dissections for other m. For example, Xia and Yao give the 4-dissection of $Q(t,m)$ in the case $4|m$, by using properties of Ramanujan's theta function $f(a,b) = (-a, -b, ab; ab)_\infty$.

Further, not all products for which dissections have been given fit the pattern of $Q(t,m)$. For example, Hirschhorn and Roselin [15] have given the 2-, 3-, 4- and 6-dissections of Ramanujans cubic continued fraction $RC(q)$ (see (1.5)).

Theorem 3.1. *Let* $m \geq 5$ *be an integer relatively prime to 6 and let* $t \in \{1, 2, \ldots, (m-1)/2\}$ *such that* $\gcd(t,m) = 1$*. Let* $Q(t,m)$ *be as defined at* (1.1).
(i)If $m \equiv 1 \pmod 6$*, then*

(3.1)

$$Q(t,m)=\sum_{r=0}^{\frac{m-1}{3}}(-1)^r q^{\frac{1}{2}r(m(3r-1)+6t)}Q\left(\frac{1}{6}m\left(m^2+m(6r-1)+6t\right),m^3\right)$$

$$+\sum_{r=0}^{\frac{m-4}{3}}(-1)^r q^{\frac{1}{6}(m-3r-2)\left(m^2-m(3r+1)-6t\right)}Q\left(\frac{1}{2}m\left(m^2-m(2r+1)-2t\right),m^3\right)$$

$$+\sum_{r=0}^{\frac{m-7}{6}}(-1)^r q^{\frac{1}{2}(3r+1)(mr+2t)}Q\left(\frac{1}{6}m(m(m-6r-1)-6t),m^3\right)$$

$$+\sum_{r=0}^{\frac{m-7}{6}}(-1)^{\frac{1}{6}(m+6r+11)}q^{\frac{1}{24}(m-6r-1)\left(m^2-6mr+m-12t\right)}Q\left(m(mr+t),m^3\right).$$

(ii) If $m\equiv 5 \pmod 6$, *then*

(3.2)

$$Q(t,m)=\sum_{r=0}^{\frac{m-5}{6}}(-1)^r q^{\frac{1}{2}r(m(3r-1)+6t)}Q\left(\frac{1}{6}m\left(m^2-6mr+m-6t\right),m^3\right)$$

$$+\sum_{r=0}^{\frac{m-5}{6}}(-1)^{\frac{1}{6}(m+6r+1)}q^{\frac{1}{24}(m-6r-1)\left(m^2-6mr+m-12t\right)}Q\left(m(mr+t),m^3\right)$$

$$+\sum_{r=0}^{\frac{m-2}{3}}(-1)^r q^{\frac{1}{2}(3r+1)(mr+2t)}Q\left(\frac{1}{6}m\left(m^2+6mr+m+6t\right),m^3\right)+$$

$$\sum_{r=0}^{\frac{m-5}{3}}(-1)^{r+1}q^{\frac{1}{6}(m-3r-2)(m(m-3r-1)-6t)}Q\left(\frac{1}{2}m(m(m-2r-1)-2t),m^3\right).$$

Proof. We first apply (1.10) to the two products on the left side of (1.8) to get that

$$\langle -z^3q;q^3\rangle_\infty=\sum_{r=0}^{m-1}z^{3r}q^{(3r^2-r)/2}\left\langle -z^{3m}q^{(3m^2-m)/2+3mr};q^{3m^2}\right\rangle_\infty, \tag{3.3}$$

$$z\langle -z^3q^2;q^3\rangle_\infty=\sum_{r=0}^{m-1}z^{3r+1}q^{(3r^2+r)/2}\left\langle -z^{3m}q^{(3m^2+m)/2+3mr};q^{3m^2}\right\rangle_\infty.$$

The next step is to replace q with q^m and then set $z=-q^t$ to get that

$$Q(t,m)=\sum_{r=0}^{m-1}\left[a_r+b_r\right], \tag{3.4}$$

where

$$a_r=(-1)^r q^{mr(3r-1)/2+3rt}\left\langle q^{m^2(3m-6r+1)/2-3mt};q^{3m^3}\right\rangle_\infty, \tag{3.5}$$

$$b_r = (-1)^r q^{mr(3r+1)/2+(3r+1)t} \left\langle q^{m^2(3m-6r-1)/2-3mt}; q^{3m^3} \right\rangle_\infty .$$

The next step is to combine pairs of terms containing powers of q in the same arithmetic progression modulo m, using the quintuple product identity. It is easy to see that all the powers of q in the series expansion of each of a_r and b_r lie in a single arithmetic progression modulo m. Further, it can be seen that, in the case of a_r, which arithmetic progression is determined completely by the value of $3rt \pmod{m}$, where the $3rt$ comes from the exponent of the power of q in front of the Jacobi triple product. Likewise the powers of q in the series expansion of b_r lie in the arithmetic progression determined by $(3r+1)t \pmod{m}$.

Since m is relatively prime to 3 and t is an integer, $1 \le t \le (m-1)/2$ such that $\gcd(t, m) = 1$, then if

$$\begin{aligned} A &= \{3rt : 0 \le r \le (m-1)\} \pmod{m}, \\ B &= \{(3r+1)t : 0 \le r \le (m-1)\} \pmod{m}, \end{aligned}$$

then

$$A = B = \{0, 1, 2, \dots, m-2, m-1\},$$

and each term in the sums at (3.1) and (3.2) will contain powers of q in a distinct arithmetic progressions modulo m, and all m arithmetic progressions will be represented.

Thus each term in the set $\{a_r\}_{r=0}^{m-1}$ may be matched with a term in the set $\{b_r\}_{r=0}^{m-1}$, such that the series expansion of both terms contain just powers of q in the same arithmetic progression modulo m.

To determine this matching pattern, we examine the situation for some prime values of m, and then it will be seen that this matching pattern will carry over to the case where m is composite.

The pattern of this matching for primes $\equiv 1 \pmod{6}$ is different to that for primes $\equiv 5 \pmod{6}$. Suppose that for a certain value of t, some a_r is matched with a certain b_s, so that $3rt \equiv (1+3s)t \pmod{m}$. Since $1 \le t \le (m-1)/2$ and $\gcd(t, m) = 1$, then $3r \equiv (1+3s) \pmod{m}$, and thus each such pair (r, s) is the same for all values of t, and thus it is enough to consider the case $t = 1$. We first consider primes $\equiv 1 \pmod{6}$, and take $m = 19$ as an example. The entries in the body of the table are reduced modulo $m = 19$.

r	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$3r$	**0**	**3**	**6**	**9**	**12**	**15**	**18**	**2**	**5**	**8**	**11**	**14**	**17**	1	4	7	10	13	16
$3r+1$	1	4	7	10	13	16	**0**	**3**	**6**	**9**	**12**	**15**	**18**	**2**	**5**	**8**	**11**	**14**	**17**

The content of the above table is meant as illustration/aid in understanding the following analysis. If $3r \equiv 3s+1 \pmod{m}$, then $3(r-s) \equiv 1 \pmod{m}$. Also, given that $0 \le r, s \le m-1$ it follows that

$$-3m + 3 \le 3(r-s) \le 3m - 3.$$

Upon putting these bounds together with the previous congruence and recalling that we are now considering the case $m \equiv 1 \pmod 6$, we have that either $3(r-s) = -m+1$ or $3(r-s) = 2m+1$.

From the first equation (this pairing is shown in bold in the table)

$$s = \frac{m-1}{3} + r \leq m-1 \Longrightarrow 0 \leq r \leq \frac{2(m-1)}{3}.$$

From the second equation (this pairing is shown in regular font in the table)

$$r = \frac{2m+1}{3} + s \leq m-1 \Longrightarrow 0 \leq s \leq \frac{m-4}{3}.$$

Hence terms are combined as follows:

$$\begin{aligned} &a_r + b_{(m-1)/3+r}, && 0 \leq r \leq \frac{2(m-1)}{3}, \\ &a_{(2m+1)/3+s} + b_s, && 0 \leq s \leq \frac{m-4}{3}. \end{aligned}$$

In what follows, we again allow $m \equiv 1 \pmod 6$ to be composite, and using the same matching works to allow terms to be combined via the quintuple product identity, as it is clear that nothing in the next few paragraphs is dependent upon m being prime.

It is a straightforward (if slightly tedious) check that

$$\begin{aligned} a_r + b_{(m-1)/3+r} &= (-1)^r q^{r(m(3r-1)/2+3t)} \Bigg[\left\langle q^{\frac{1}{2}m\left(3m^2-6rm+m-6t\right)}; q^{3m^3} \right\rangle_\infty \\ &\quad + q^{\frac{1}{6}m\left(m^2+m(6r-1)+6t\right)} \left\langle q^{\frac{1}{2}m\left(m^2-6rm+m-6t\right)}; q^{3m^3} \right\rangle_\infty \Bigg] \\ &= (-1)^r q^{r(m(-1+3r)/2+3t)} Q\left(\frac{1}{6}m(m^2+m(-1+6r)+6t); m^3\right) \end{aligned}$$

where the last equality follows from the quintuple product identity (1.8) (with q replaced with q^{m^3} and $z = -q^{1/6m\left(m^2+m(6r-1)+6t\right)}$). Similarly,

$$\begin{aligned} &a_{(2m+1)/3+s} + b_s \\ &\quad = (-1)^r q^{\frac{1}{6}(m-3r-1)\left(m^2-3mr-6t\right)} \Bigg[\left\langle q^{\frac{1}{2}m\left(m^2+6rm+m+6t\right)}; q^{3m^3} \right\rangle_\infty \\ &\qquad + q^{\frac{1}{6}m\left(-m^2+6mr+m+6t\right)} \left\langle q^{\frac{1}{2}m(m(3m-6r-1)-6t)}; q^{3m^3} \right\rangle_\infty \Bigg] \\ &\quad = (-1)^r q^{1/6(-1+m-3r)(m^2-3mr-6t)} Q\left(\frac{1}{6}m(m-m^2+6mr+6t); m^3\right), \end{aligned}$$

where this time the last equality follows from the quintuple product identity (1.8) with q replaced with q^{m^3} and $z = -q^{1/6m(m-m^2+6mr+6t)}$. What has been shown at this point is that if $m \equiv 1 \pmod 6$ then

$$Q(t, m)$$

$$= \sum_{r=0}^{2(m-1)/3} (-1)^r q^{r(m(-1+3r)/2+3t)} Q\left(\frac{1}{6}m(m^2+m(-1+6r)+6t); m^3\right) + \sum_{r=0}^{(m-4)/3} (-1)^r q^{(-1+m-3r)(m^2-3mr-6t)/6} Q\left(\frac{1}{6}m(m-m^2+6mr+6t); m^3\right).$$

The problem with this representation is that both sums contain infinite products with negative powers of q for values of r that lie in the upper half of the range of summation. The reason this is a problem is that it that in this form there are minor complications when one examine sign patterns modulo m in the next theorem.

To overcome this, the first sum is split in two as follows:

$$\sum_{r=0}^{2(m-1)/3} = \sum_{r=0}^{(m-1)/3} + \sum_{r=(m+2)/3}^{2(m-1)/3},$$

then (1.11) is applied to the infinite products in the numerator of the terms in the second sum, and then the summation variable in this second sum is shifted so that it also starts at $r=0$.

Likewise, the second sum is split as

$$\sum_{r=0}^{(m-4)/3} = \sum_{r=0}^{(m-7)/6} + \sum_{r=(m-1)/6}^{(m-4)/3},$$

then (1.11) is applied to the infinite products in both the numerator and denominator of the terms in the second sum, and then the summation variable in this second sum is shifted so that it also starts at $r=0$. This completes the proof of (3.1).

The proof for integers $m \equiv 5 \pmod 6$ essentially follows the same general idea, with $m=17$ used as an illustrative example. This time the entries in the table are reduced modulo 17.

r	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$3r$	**0**	**3**	**6**	**9**	**12**	**15**	1	4	7	10	13	16	2	5	8	11	14
$3r+1$	1	4	7	10	13	16	2	5	8	11	14	**0**	**3**	**6**	**9**	**12**	**15**

This time the terms are combined as follows (the first pairing is shown in bold in the table, and the second is shown in regular font):

$$\begin{aligned} &a_r + b_{(2m-1)/3+r}, && r = 0, 1, \ldots, (m-2)/3, \\ &a_{(m+1)/3+r} + b_r, && r = 0, 2, \ldots, 2(m-2)/3. \end{aligned}$$

Here the sums are split as

$$\sum_{r=0}^{(m-2)/3} = \sum_{r=0}^{(m-5)/6} + \sum_{r=(m+1)/6}^{(m-2)/3},$$

$$\sum_{r=0}^{2(m-2)/3} = \sum_{r=0}^{(m-2)/3} + \sum_{r=(m+1)/3}^{2(m-2)/3}.$$

The sums in the upper half of each summation range are treated similarly to the sums in the upper half of each summation range in the $m \equiv 1 \pmod 6$ case, resulting in the expansion at (3.2). $\square$

The 5-dissections of Ramanujan's function

$$R(q) := \frac{(q^2, q^3; q^5)_\infty}{(q, q^4; q^5)_\infty} \tag{3.6}$$

and it reciprocal, which were proven by Hirschhorn [11], follow from Theorem 3.1.

We remark that one difference between the method of proof employed by Hirschhorn in proving the results in Corollary 3.1 and the method employed by the present author in proving Theorem 3.1 is that Hirschhorn worked with the series form of the quintuple product identity (1.8) and produced the 5-dissection of the resulting series, before recombining the resulting series into quintuple products. By contrast, here we worked with the triple product form of the quintuple product identity throughout, producing m-dissections of Jacobi triple products and then recombining pairs of triple products into quintuple products.

Corollary 3.1. *If $|q| < 1$, then*

$$\begin{aligned} R(q) = \frac{\big(q^{125}; q^{125}\big)_\infty}{(q^5; q^5)_\infty} \Bigg[&\frac{\big(q^{40}, q^{85}; q^{125}\big)_\infty}{\big(q^{20}, q^{105}; q^{125}\big)_\infty} + q \frac{\big(q^{60}, q^{65}; q^{125}\big)_\infty}{\big(q^{30}, q^{95}; q^{125}\big)_\infty} \\ &- q^7 \frac{\big(q^{35}, q^{90}; q^{125}\big)_\infty}{\big(q^{45}, q^{80}; q^{125}\big)_\infty} - q^3 \frac{\big(q^{10}, q^{115}; q^{125}\big)_\infty}{\big(q^5, q^{120}; q^{125}\big)_\infty} - q^{14} \frac{\big(q^{15}, q^{110}; q^{125}\big)_\infty}{\big(q^{55}, q^{70}; q^{125}\big)_\infty} \Bigg], \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} R(q)^{-1} = \frac{\big(q^{125}; q^{125}\big)_\infty}{(q^5; q^5)_\infty} \Bigg[&\frac{\big(q^{30}, q^{95}; q^{125}\big)_\infty}{\big(q^{15}, q^{110}; q^{125}\big)_\infty} - q \frac{\big(q^{20}, q^{105}; q^{125}\big)_\infty}{\big(q^{10}, q^{115}; q^{125}\big)_\infty} \\ &+ q^2 \frac{\big(q^{55}, q^{70}; q^{125}\big)_\infty}{\big(q^{35}, q^{90}; q^{125}\big)_\infty} - q^{18} \frac{\big(q^5, q^{120}; q^{125}\big)_\infty}{\big(q^{60}, q^{65}; q^{125}\big)_\infty} - q^4 \frac{\big(q^{45}, q^{80}; q^{125}\big)_\infty}{\big(q^{40}, q^{85}; q^{125}\big)_\infty} \Bigg]. \end{aligned} \tag{3.8}$$

Proof. These follow directly from the $m = 5$ case of (3.2) upon setting $t = 1$ and $t = 2$ respectively, and dividing both sides by $(q^5; q^5)_\infty$. $\square$

As an example of a dissection we believe to be new, we give the 7-dissection of the product $Q(1, 7)/(q^7; q^7)_\infty$, which follows from setting $m = 7$ and $t = 1$ in (3.1), and then dividing both sides by $(q^7; q^7)_\infty$.

Example 2.

(3.9)

$$\frac{\big(q^2, q^5; q^7\big)_\infty}{\big(q, q^6; q^7\big)_\infty} = \frac{\big(q^{343}; q^{343}\big)_\infty}{\big(q^7; q^7\big)_\infty} \Bigg[\frac{\big(q^{112}, q^{231}; q^{343}\big)_\infty}{\big(q^{56}, q^{287}; q^{343}\big)_\infty} + q \frac{\big(q^{84}, q^{259}; q^{343}\big)_\infty}{\big(q^{42}, q^{301}; q^{343}\big)_\infty}$$

$$+ q^{30}\frac{(q^{63},q^{280};q^{343})_\infty}{(q^{140},q^{203};q^{343})_\infty} - q^{10}\frac{(q^{133},q^{210};q^{343})_\infty}{(q^{105},q^{238};q^{343})_\infty} - q^{11}\frac{(q^{14},q^{329};q^{343})_\infty}{(q^{7},q^{336};q^{343})_\infty} - q^{5}\frac{(q^{161},q^{182};q^{343})_\infty}{(q^{91},q^{252};q^{343})_\infty} + q^{41}\frac{(q^{35},q^{308};q^{343})_\infty}{(q^{154},q^{189};q^{343})_\infty}\Bigg].$$

Hirschhorn [11] was able to use the 5-dissections in Corollary 3.1 to prove an observation of Richmond and Szekeres [19] about the eventual non-vanishing and 5-periodicity of the signs of the coefficients in the series expansion of $R(q)$ and $R(q)^{-1}$. A combinatorial proof was given by Andrews [1]. Both of these (non-vanishing and 5-periodicity) follow from writing

$$(q^5,q^5)_\infty = (q^5,q^{10},q^{15},\dots,q^{120},q^{125};q^{125})_\infty,$$

cancelling the products in the various numerators in the 5-dissections of $R(q)$ and $R(q)^{-1}$, and then interpreting each of the resulting products as generating functions for certain kinds of restricted partition functions (with some parts coming in two flavors). Similar reasoning applied to the m-dissections produced in Theorem 3.1 allows a similar statement to be proved about the series expansion of any product of the form

$$R_{m,t}(q) := \frac{(q^{2t},q^{m-2t};q^m)_\infty}{(q^t,q^{m-t};q^m)_\infty} = \frac{Q(t,m)}{(q^m;q^m)_\infty}, \tag{3.10}$$

where $m \geq 5$ is an integer relatively prime to 6 and t is an integer in the interval $1 \leq t \leq (m-1)/2$. In fact it is quite easy, for each such product, to state explicitly what the pattern of $+/-$ signs is, and also to state explicitly where the last vanishing coefficient occurs. The main idea behind the proofs of these statements may be seen by examining Hirschhorn's 5-dissection of $R(q) = R_{5,1}$ at (3.7) above. If all the infinite q-products are dropped from the right side, one gets the polynomial

$$f_{5,1}(q) := 1 + q - q^7 - q^3 - q^{14}.$$

This tells us that the last vanishing coefficient is that multiplying $q^{14-5} = q^9$. Next, the exponents are reduced modulo 5, one gets the polynomial

$$g_{5,1}(q) := 1 + q - q^2 - q^3 - q^4.$$

This tells us that the sign of the coefficient multiplying q^{5n+j} for $n \geq 2$ and $j = 0,1,2,3,4$ is, respectively, $+,+,-,-,-$.

Likewise, it is now possible to drop the infinite q-products in Theorem 3.1 to derive polynomials which predict the sign pattern in the series expansion of the product $R_{m,t}(q)$ at (3.10), and also to determine explicitly the exact order of the last vanishing term. We restrict to the case where t is relatively prime to m, so that, as remarked above, each term in the sums at (3.1) and (3.2) will contain powers of q in a distinct arithmetic progressions modulo m, and all m arithmetic progressions will be present.

If $\gcd(t,m) = d > 1$, then by setting $t_1 := t/d$ and $m_1 := m/d$ it can be seen that m_1 is also relatively prime to 6, $\gcd(t_1,m_1) = 1$, and that $R(t,m)$

may be derived from $R(t_1, m_1)$ (for which information may be acquired using Theorem 3.2) simply as a consequence of the dilation $q \to q^d$.

Theorem 3.2. *Let $m \geq 5$ be an integer relatively prime to 6, and let $t \in \{1, 2, \dots, (m-1)/2\}$ be such that $\gcd(t, m) = 1$. Define the function $R_{m,t}(q)$ and the sequence c_n by*

$$R_{m,t}(q) := \frac{(q^{2t}, q^{m-2t}; q^m)_\infty}{(q^t, q^{m-t}; q^m)_\infty} = \frac{Q(t,m)}{(q^m; q^m)_\infty} =: \sum_{n=0}^{\infty} c_n q^n. \tag{3.11}$$

In both the cases $m \equiv 1 \pmod 6$ and $m \equiv 5 \pmod 6$, the largest value of n for which $c_n = 0$ is

$$n = n_0 := \begin{cases} \frac{1}{6}(m-4)m(m+1) + (m-1)t, & 1 \leq t < \frac{m-1}{2}, \\ \\ \frac{1}{6}(m-4)m(m+1) + (m-1)t + 2m, & t = \frac{m-1}{2}. \end{cases} \tag{3.12}$$

If $m \equiv 1 \pmod 6$, define

$$f_{m,t}(q) = \sum_{r=0}^{\frac{m-1}{3}} (-1)^r q^{\frac{1}{2}r(m(3r-1)+6t)} + \sum_{r=0}^{\frac{m-4}{3}} (-1)^r q^{\frac{1}{6}(m-3r-2)\left(m^2-m(3r+1)-6t\right)} + \sum_{r=0}^{\frac{m-7}{6}} (-1)^r q^{\frac{1}{2}(3r+1)(mr+2t)} + \sum_{r=0}^{\frac{m-7}{6}} (-1)^{\frac{1}{6}(m+6r+11)} q^{\frac{1}{24}(m-6r-1)\left(m^2-6mr+m-12t\right)}. \tag{3.13}$$

If $m \equiv 5 \pmod 6$, define

$$f_{m,t}(q) = \sum_{r=0}^{\frac{m-5}{6}} (-1)^r q^{\frac{1}{2}r(m(3r-1)+6t)} + \sum_{r=0}^{\frac{m-5}{6}} (-1)^{\frac{1}{6}(m+6r+1)} q^{\frac{1}{24}(m-6r-1)\left(m^2-6mr+m-12t\right)} + \sum_{r=0}^{\frac{m-2}{3}} (-1)^r q^{\frac{1}{2}(3r+1)(mr+2t)} + \sum_{r=0}^{\frac{m-5}{3}} (-1)^{r+1} q^{\frac{1}{6}(m-3r-2)(m(m-3r-1)-6t)}. \tag{3.14}$$

In either case, let

$$g_{m,t}(q) := \sum_{j=0}^{m-1} d_j q^j \tag{3.15}$$

be the polynomial derived from $f_{m,t}(q)$ by reducing all the exponents modulo m. If $mk + j > n_0$, where k is some integer and j is an integer such that $0 \leq j \leq m-1$, then $c_{mk+j} \neq 0$ and

$$sign(c_{mk+j}) = sign(d_j). \tag{3.16}$$

Proof. As in the proof of Theorem 2.1, observe that for each of the cases $m \equiv 1 \pmod 6$ and $m \equiv 5 \pmod 6$, that $f_{m,t}(q)$ is derived from the corresponding m-dissection in Theorem 3.1 by setting each $Q(\_,\_) = 1$.

One difference from Theorem 2.1 is that $f_{m,t}(q)$, in contrast to $f_m(q)$ (which contained $\sim m/2$ distinct powers of q) contains m distinct powers of q (recall that the exponents of the powers of q in front of the $Q(\_,\_)$ in the corresponding m-dissections in Theorem 3.1 are all distinct modulo m by construction). Since all of the coefficients in $f_{m,t}(q)$ have the form $(-1)^u$ for some integer u, then $g_{m,t}(q)$ has degree $m-1$ and each $d_j \in \{-1, 1\}$.

Next we consider the case $m \equiv 1 \pmod 6$ and examine an arbitrary term in the m-dissection of the quotient $Q(t,m)/(q^m;q^m)_\infty$ derived from (3.1) (similar reason applies to the case $m \equiv 5 \pmod 6$ and (3.2), so the proof in this case is omitted). We consider a term coming from the third sum in (3.1) (similar analysis applies to terms coming from each of the other three sums):

$$(3.17)\quad (-1)^r q^{\frac{1}{2}(3r+1)(mr+2t)} \frac{Q\left(a, m^3\right)}{(q^m;q^m)_\infty} = (-1)^r q^{\frac{1}{2}(3r+1)(mr+2t)} \frac{(q^{2a}, q^{m^3-2a}, q^{m^3}; q^{m^3})_\infty}{(q^a, q^{m^3-a}; q^{m^3})_\infty (q^m;q^m)_\infty},$$

where for ease of notation we have written a for $m(m(m-6r-1)-6t)/6$. By an argument that is essentially the same as that employed in the proof of Theorem 2.2, it can be shown that if the infinite product in (3.17) is expanded as an infinite series in q^m, then all coefficients of powers of q^m are non-zero.

It can thus be seen that in the series expansion of $Q(t,m)/(q^m;q^m)_\infty$, all terms in the sub-sequence $\{c_{mk+(3r+1)t}\}$ are non-zero for $k \geq r(3r+1)/2$ and all have sign $(-1)^r$, and all terms vanish for $1 \leq k < r(3r+1)/2$. A similar statement holds for each of the other terms in $f_{m,t}(q)$, apart from in the exceptional case when $t = (m-1)/2$.

When $t = (m-1)/2$, then the $r = (m-1)/3$ term in the first sum in (3.1) and the $r = (m-2)/3$ term in the third sum at (3.2) each have the infinite product $(q^m; q^{m^3})_\infty$ in the numerator of the factor $Q(-, m^3)$, and by similar reasoning to that used in the proof of Theorem 2.2, the series expansion of the quotient $Q(-,m^3)/(q^m;q^m)_\infty$ has a zero coefficient for q^m, and this is the only power of q^m with a zero coefficient.

Let M denote the exponent of the greatest power of q in $f_{m,t}(q)$. It turns out that this M comes from the terms that lead to the exceptional cases described in the previous paragraph, namely, $r = (m-1)/3$ term in the first sum in (3.1) and the $r = (m-2)/3$ term in the third sum at (3.2). It is not difficult to see that this largest power must occur for either the upper or lower value of r in one of the four sums, and by comparing all 8 possible values it is found that it does occur for the values of r stated, and that

$$M=\frac{(m-1)(m^2-2m+6t)}{6}.$$

From what has just been shown, it can be seen that when $t\neq(m-1)/2$ the largest value of n for which $c_n=0$ is

$$n_0=M-m=\frac{1}{6}(m-4)m(m+1)+(m-1)t,$$

and that $n_0=M+m$ when $t=(m-1)/2$, as claimed. The claim at (3.16) now follows from what has already been shown above.

As already mentioned above, the proofs of the statements for $m\equiv 5 \pmod 6$ are similar, and are omitted. $\square$

As an example of the information the previous theorem provides, consider the following example.

Example 3. *Let*

$$R_{19,3}(q):=\frac{(q^6,q^{13};q^{19})_\infty}{(q^3,q^{16};q^{19})_\infty}=:\sum_{n=0}^{\infty}c_nq^n. \tag{3.18}$$

Upon substituting $m=19$ *and* $t=3$ *into* (3.12)*, one gets that the largest value of* n *for which* $c_n=0$ *is* $n=n_0=1004$*. After substituting* $m=19$ *and* $t=3$ *into* (3.13)*, one gets the polynomial*

$$\begin{aligned} f_{19,3}(q)&=1+q^3-q^{13}-q^{28}-q^{29}-q^{50}+q^{80}+q^{113}+q^{115}\\ &+q^{154}-q^{204}-q^{255}-q^{258}+q^{385}+q^{454}-q^{623}-q^{710}+q^{918}+q^{1023}. \end{aligned} \tag{3.19}$$

After reducing the exponents in $f_{19,3}(q)$ *modulo 19, one gets the polynomial*

$$\begin{aligned} g_{19,3}(q)&=1+q+q^2+q^3+q^4+q^5+q^6-q^7-q^8-q^9\\ &-q^{10}-q^{11}-q^{12}-q^{13}-q^{14}-q^{15}+q^{16}+q^{17}+q^{18}. \end{aligned} \tag{3.20}$$

Thus the signs of the coefficients c_n*, for* $n>1004$*, in the arithmetic progressions* $0 \pmod{19}$, $1 \pmod{19}$, $\dots$ $18 \pmod{19}$*, respectively, are*

$$+,+,+,+,+,+,+,-,-,-,-,-,-,-,-,-,+,+,+,$$

respectively.

A second example illustrates the exceptional case when $t=(m-1)/2$.

Example 4. *Let*

$$R_{17,8}(q):=\frac{(q^{16},q;q^{17})_\infty}{(q^8,q^9;q^{17})_\infty}=:\sum_{n=0}^{\infty}c_nq^n. \tag{3.21}$$

Upon substituting $m=17$ *and* $t=8$ *into the* $t=(m-1)/2$ *case of* (3.12)*, one gets that the largest value of* n *for which* $c_n=0$ *is* $n=n_0=825$*. After* *substituting* $m = 17$ *and* $t = 8$ *into* (3.14), *one gets the polynomial*

$$f_{17,8}(q) = 1 - q + q^8 - q^{10} - q^{41} + q^{45} - q^{66} + q^{71} + q^{133} - q^{140} + q^{175} - q^{183} - q^{335} + q^{346} + q^{546} - q^{560} - q^{808}. \tag{3.22}$$

After reducing the exponents in $f_{17,8}(q)$ *modulo 17, one gets the polynomial*

$$g_{17,8}(q) = 1 - q + q^2 + q^3 - q^4 + q^5 + q^6 - q^7 + q^8 - q^9 - q^{10} + q^{11} - q^{12} - q^{13} + q^{14} - q^{15} - q^{16}. \tag{3.23}$$

Thus the signs of the coefficients c_n, *for* $n > 825$, *in the arithmetic progressions* $0 \pmod{17}$, $1 \pmod{17}$, $\dots$ $16 \pmod{17}$, *respectively, are*

$$+, -, +, +, -, +, +, -, +, -, -, +, -, -, +, -, -,$$

respectively.

Remark: These examples also permit us to point out another property of the polynomials $f_{m,t}(q)$, namely that the terms encode more precise information about the signs of the coefficients in each arithmetic progression than does $g_{m,t}(q)$. (As indicated in the proof of Theorem 3.2, the situation is slightly different for the largest exponent M in $f_{m,t}(q)$ when $t = (m-1)/2$, in that the last zero coefficient in that arithmetic progression is that multiplying q^{M+m}.)

For example, consider the term $+q^{346}$ in $f_{17,8}(q)$ in Example 4. Since $346 = 17(20) + 6$, then $c_{17k+6} = 0$ for $k < 20$, and $c_{17k+6} > 0$ for $k \geq 20$. In contrast, the term $+q^6$ in $g_{17,8}(q)$ just tells us that the coefficients c_{17k+6} are all positive from "some" point on.

4. Lambert Series identities deriving from $(q;q)_\infty^3$

Once an m-dissection of $(q;q)_\infty$ has been derived, then clearly an m-dissection of $(q;q)_\infty^3$ can be found simply by multiplying out the cube of the m-dissection of $(q;q)_\infty$ and collecting together products that give terms in the same arithmetic progression.

However, it is also possible to give an alternative m-dissection of $(q;q)_\infty^3$ in terms of Lambert series, and hence derive expressions for the individual Lambert series in terms of infinite products by comparing corresponding terms in the two expansions. We first derive the m-dissections in terms of Lambert series.

Remark: For m even, the next theorem gives an $m/2$ - dissection, not an m-dissection.

Theorem 4.1. *Let* $|q| < 1$.

(i) If $m \geq 3$ is an odd positive integer, then

$$(q;q)_\infty^3 = (-1)^{(m-1)/2} m q^{(m^2-1)/8} (q^{m^2};q^{m^2})_\infty^3 + \sum_{r=1}^{(m-1)/2} (-1)^{r+1} q^{r(r-1)/2} \left\langle q^{m(m-1+2r)/2}; q^{m^2} \right\rangle_\infty \times \left[2r-1+2m\sum_{k=1}^{\infty} \frac{q^{km(m+1-2r)/2}}{1-q^{km^2}} - 2m\sum_{k=1}^{\infty} \frac{q^{km(m-1+2r)/2}}{1-q^{km^2}} \right]. \tag{4.1}$$

(ii) If $m \geq 2$ is an even positive integer, then

$$(q;q)_\infty^3 = \sum_{r=1}^{m/2} (-1)^{r+1} q^{r(r-1)/2} \left\langle -q^{m(m-1+2r)/2}; q^{m^2} \right\rangle_\infty \times \left[2r-1+2m\sum_{k=1}^{\infty} \frac{(-1)^k q^{km(m+1-2r)/2}}{1-q^{km^2}} - 2m\sum_{k=1}^{\infty} \frac{(-1)^k q^{km(m-1+2r)/2}}{1-q^{km^2}} \right]. \tag{4.2}$$

Proof. It can be seen that dividing the left side of (1.10) by $1-z$ and then letting $z \to 1$ leads to the left sides of (4.1) and (4.2), and it will be shown that performing the same operations on the right side of (1.10), in the cases m is odd and m is even, will lead, respectively, to the right sides of (4.1) and (4.2).

Proof of (4.1). In the case of (4.1), it will be shown, for $2 \leq r \leq (m-1)/2$, that adding the r-th term and $m+1-r$-th term in the sum on the right side of (1.10), dividing the resulting sum by $1-z$ and letting $z \to 1$ leads to the r-th term in the sum on the right side of (4.1). Likewise, adding the $r=0$ and the $r=1$ terms in the sum on the right side of (1.10), dividing the sum by $1-z$ and letting $z \to 1$ leads to the $r=1$ term in the sum on the right side of (4.1). Finally, in the case m is odd, dividing the unmatched $r=(m+1)/2$-th term in the sum on the right side of (1.10) by $1-z$ and letting $z \to 1$ leads to the term before the sum on the right side of (4.1).

For $2 \leq r \leq (m-1)/2$, the $m-r+1$-th term in the sum at (1.10) is

$$(-1)^r q^{\left(m^2-2mr+m+(r-1)r\right)/2} z^{m-r+1} \left\langle q^{-m(m-2r+1)/2} z^{-m}; q^{m^2} \right\rangle_\infty = (-1)^{r+1} q^{r(r-1)/2} z^{1-r} \left\langle q^{m(m+2r-1)/2} z^{-m}; q^{m^2} \right\rangle_\infty.$$

The equality above follows from applying transformations like those stated at (1.11) above. Hence, adding the r-th term and $m+1-r$-th term in the sum on the right side of (1.10) gives

$$(-1)^r q^{r(r-1)/2} \left[z^r \left\langle q^{m(m-2r+1)/2} z^{-m}; q^{m^2} \right\rangle_\infty - z^{1-r} \left\langle q^{m(m+2r-1)/2} z^{-m}; q^{m^2} \right\rangle_\infty \right].$$

That dividing by $1-z$ and taking the limit as $z \to 1$ leads to the r-th term in the sum on the right side of (4.1) follows as a consequence of L'Hospital's rule and logarithmic differentiation.

That adding the $r=0$ and $r=1$ terms in the sum on the right side of (1.10), dividing the sum by $1-z$ and letting $z \to 1$ leads to the $r=1$ term in the sum on the right side of (4.1) follows by similar arguments.

The remaining claim above about the unmatched $r=(m+1)/2$-th term in the sum on the right side of (1.10) leading to the term before the sum on the right side of (4.1) follows after writing $(z^{-m};q^{m^2})_\infty$ in the $r=(m+1)/2$-th term as $(1-z^{-m})(z^{-m}q^{m^2};q^{m^2})_\infty$ before dividing by $1-z$ and taking the limit as $z \to 1$.

The proof of (4.2) is similar, and so is omitted. □

The Lambert series identities in the next corollary follow from Theorems 2.1 and 4.1. They were first proven by Hirschhorn [12], who showed that they followed from the pair of identities in Corollary 4.3.

Corollary 4.1. *With the usual notation, let $f_i := (q^i;q^i)_\infty$ and let $R=R(q)$ be as defined at* (3.6). *Then*

$$1+10\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{5n}}-10\sum_{n=1}^{\infty}\frac{q^{3n}}{1-q^{5n}}=\left(\frac{1}{R^3}-3qR^2\right)\frac{f_5^2}{(q^2,q^3;q^5)_\infty},\tag{4.3}$$

and

$$3+10\sum_{n=1}^{\infty}\frac{q^{n}}{1-q^{5n}}-10\sum_{n=1}^{\infty}\frac{q^{4n}}{1-q^{5n}}=\left(\frac{3}{R^2}+qR^3\right)\frac{f_5^2}{(q,q^4;q^5)_\infty}.\tag{4.4}$$

Proof. For ease of notation, let $R_5 := R(q^5)$, so that by (2.6),

$$(q;q)_\infty = f_{25}\left[\frac{1}{R_5}-q-q^2R_5\right].$$

Hence

$$(q;q)_\infty^3 = f_{25}^3\left[\left(\frac{1}{R_5^3}-3q^5R_5^2\right)-q\left(\frac{3}{R_5^2}+q^5R_5^3\right)+5q^3\right]\tag{4.5}$$

The identities at (4.3) and (4.4) now follow by comparing the right side of (4.5) with the right side of the $m=5$ case of (4.1), equating the parts of each 5-dissection containing powers of q congruent to 0 (mod 5) and 1 (mod 5) respectively (and cancelling a factor of q in the latter case), and finally replacing q^5 with q. □

Similar identities may be derived from considering 7-dissections.

Corollary 4.2. *As above, let $f_i := (q^i;q^i)_\infty$ and define*

$$A=A(q):=\frac{(q^2,q^5;q^7)_\infty}{(q,q^6;q^7)_\infty},\ B=B(q):=\frac{(q^3,q^4;q^7)_\infty}{(q^2,q^5;q^7)_\infty}.\tag{4.6}$$

Then
(4.7)
$$1+14\sum_{n=1}^{\infty}\frac{q^{3n}}{1-q^{7n}}-14\sum_{n=1}^{\infty}\frac{q^{4n}}{1-q^{7n}}=\left(A^3+\frac{3Bq}{A}-\frac{6q}{B}\right)\frac{f_7^2}{(q^3,q^4;q^7)_\infty},$$
(4.8)
$$3+14\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{7n}}-14\sum_{n=1}^{\infty}\frac{q^{5n}}{1-q^{7n}}=\left(3A^2B-\frac{6q}{A}-\frac{q^2}{A^3B^3}\right)\frac{f_7^2}{(q^2,q^5;q^7)_\infty},$$
(4.9)
$$5+14\sum_{n=1}^{\infty}\frac{q^{n}}{1-q^{7n}}-14\sum_{n=1}^{\infty}\frac{q^{6n}}{1-q^{7n}}=\left(6AB-B^3+\frac{3q}{AB^2}\right)\frac{f_7^2}{(q,q^6;q^7)_\infty},$$
and
$$A^3B-A^2B^3=q. \tag{4.10}$$

Proof. For ease of notation, let $A_7 := A(q^7)$ and $B_7 := B(q^7)$, so that by (2.6),
$$(q;q)_\infty=f_{49}\left[A_7-B_7q-q^2+\frac{q^5}{A_7B_7}\right].$$
Then
$$\begin{aligned}(q;q)_\infty^3=f_{49}^3\Bigg[&\left(\frac{3B_7q^7}{A_7}+A_7^3-\frac{6q^7}{B_7}\right)+q\left(\frac{q^{14}}{A_7^3B_7^3}-3A_7^2B_7+\frac{6q^7}{A_7}\right)\\&+q^2\left(\frac{3q^7}{A_7B_7}+3A_7B_7^2-3A_7^2\right)+q^3\left(\frac{3q^7}{A_7B_7^2}+6A_7B_7-B_7^3\right)\\&+q^4\left(-\frac{3q^7}{A_7^2B_7}+3A_7-3B_7^2\right)+q^5\left(-\frac{3q^7}{A_7^2B_7^2}+\frac{3A_7}{B_7}-3B_7\right)-7q^6\Bigg].\end{aligned} \tag{4.11}$$
The remainder of the proof is similar to the proof of Corollary 4.1, and follows from comparing the expression above with the $m=7$ case of (4.1). The identity at (4.10) follows since the 7-dissection of $(q;q)_\infty^3$ given by the right side of $m=7$ case of (4.1) contains no powers of q with exponent $\equiv 2,4$ or $5 \pmod 7$. The details of the remainder of the proofs are similar to those in the previous corollary, and are omitted. □

Remark: Sets of similar Lambert series identities may likewise be derived for any prime $p\geq 5$, although the infinite product side of each such identity gets increasingly more complicated as the prime p gets larger.

By considering the m-dissection of the series in "Jacobi's identity"
$$\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}=(q;q)_\infty^3, \tag{4.12}$$
other representations for the individual terms in the m-dissection of $(q;q)_\infty^3$ may be derived. The identities in the following two corollaries, which follow from the 5-dissection and 7-dissection respectively, are given as illustrations.

The identities in Corollary 4.3 were stated by Ramanujan in the lost notebook (see [2, **Entry 1.3.1**, pages 18-21]). The earliest proof that we are aware of is that of Hirschhorn [12]. We were initially unaware of Hirschhorn's proof, and his method is essentially the same as that in the present paper. Proofs of these identities were also given by Berndt, Huang, Sohn and Son in [4]. The author was initially also unaware of these proofs, and again their method of proof is essentially the same as ours, except they work with the bilateral version of (4.12), and have $q^{1/5}$ where we have q.

Corollary 4.3. *If* $|q| < 1$, *then*

$$\sum_{k=0}^{\infty}(-1)^k\left((10k+1)q^{\frac{5k^2}{2}+\frac{k}{2}}+(10k+9)q^{\frac{5k^2}{2}+\frac{9k}{2}+2}\right)=f_5^3\left(\frac{1}{R^3}-3qR^2\right), \tag{4.13}$$

$$\sum_{k=0}^{\infty}(-1)^k\left((10k+3)q^{\frac{5k^2}{2}+\frac{3k}{2}}+(10k+7)q^{\frac{5k^2}{2}+\frac{7k}{2}+1}\right)=f_5^3\left(\frac{3}{R^2}+qR^3\right). \tag{4.14}$$

Proof. These identities follow directly from comparing the 5-dissection at (4.5) with the 5-dissection of the series at (4.12). The details are omitted. □

Corollary 4.4. *Let* $A = A(q)$ *and* $B = B(q)$ *be as defined at* (4.6). *If* $|q| < 1$, *then*

$$\sum_{k=0}^{\infty}(-1)^k\left((14k+1)q^{\frac{7k^2}{2}+\frac{k}{2}}+(14k+13)q^{\frac{7k^2}{2}+\frac{13k}{2}+3}\right)=f_7^3\left(A^3-\frac{6q}{B}+\frac{3Bq}{A}\right). \tag{4.15}$$

$$\sum_{k=0}^{\infty}(-1)^k\left((14k+3)q^{\frac{7k^2}{2}+\frac{3k}{2}}+(14k+11)q^{\frac{7k^2}{2}+\frac{11k}{2}+2}\right)=f_7^3\left(3A^2B-\frac{6q}{A}-\frac{q^2}{A^3B^3}\right). \tag{4.16}$$

$$\sum_{k=0}^{\infty}(-1)^k\left((14k+5)q^{\frac{7k^2}{2}+\frac{5k}{2}}+(14k+9)q^{\frac{7k^2}{2}+\frac{9k}{2}+1}\right)=f_7^3\left(6AB-B^3+\frac{3q}{AB^2}\right). \tag{4.17}$$

Proof. The proofs here are similar to those in the previous corollary, this time following from comparing the 7-dissection at (4.11) with the 7-dissection of the series at (4.12). Once again, the details are omitted. □

5. Lambert series identities deriving from $m-$ dissections of $(z,q/z,q;q)_\infty(qz^2,q/z^2;q^2)_\infty$

Before deriving another family of Lambert series identities, another special case of (1.9) is needed.

Corollary 5.1. *Let $|q|<1$.*
(i) If $m\geq 1$ is an odd positive integer, then

$$\left\langle q;q^2\right\rangle_\infty=\left\langle q^{m^2};q^{2m^2}\right\rangle_\infty+2\sum_{r=1}^{(m-1)/2}(-1)^rq^{r^2}\left\langle q^{m(m-2r)};q^{2m^2}\right\rangle_\infty. \tag{5.1}$$

(ii) If $m\geq 2$ is an even integer then

$$\begin{aligned}\left\langle q;q^2\right\rangle_\infty=\left\langle -q^{m^2};q^{2m^2}\right\rangle_\infty&+(-1)^{m/2}q^{m^2/4}\left\langle -1;q^{2m^2}\right\rangle_\infty\\&+2\sum_{r=1}^{m/2-1}(-1)^rq^{r^2}\left\langle -q^{m(m-2r)};q^{2m^2}\right\rangle_\infty.\end{aligned} \tag{5.2}$$

Proof. Set $z=1$ in (1.9) and observe that if

$$f_m(r):=(-1)^rq^{r^2}\left\langle(-1)^{m+1}q^{m^2-2mr};q^{2m^2}\right\rangle_\infty,$$

then $f_m(m-r)=f_m(r)$, after once again applying transformations like those stated at (1.11) above. Thus $f(r)+f(m-r)=2f(r)$, for $r=1,2,\dots,\lfloor(m-1)/2\rfloor$. The isolated q-product in (i) is $f_m(0)$, and the two isolated products in (ii) are $f_m(0)$ and $f_m(m/2)$. □

Clearly it is possible to give an m-dissection (in terms of z) of the infinite product $(z,q/z,q;q)_\infty(qz^2,q/z^2;q^2)_\infty$ in the quintuple product identity (1.8), by applying (1.9) (suitably modified) to the two triple products on the left side of (1.8). However, it can be seen from (1.9) that the m-dissection derived in this simple manner has a sum in which r runs from 0 to $m-1$, and attempting to derive Lambert series identities from these sums (dividing both sides of the m-dissection by $1-z$ and letting $z\to 1$) will lead to Lambert series that do not converge for $r>m/2$. Hence it is necessary to manipulate the terms in these m-dissections to get a sum that runs over the range $0\leq r\leq\lfloor(m-1)/2\rfloor$, as described in the next theorem.

Theorem 5.1. *Let $|q|<1$, $z\neq 0$ and $m\geq 1$ be a positive integer. Define*

(5.3)

$$\begin{aligned}\alpha_r=q^{r(3r-1)/2}\Bigg[&z^{3r}\left\langle -q^{m(3m-6r+1)/2}z^{-3m};q^{3m^2}\right\rangle_\infty\\&-z^{1-3r}\left\langle -q^{m(3m+6r-1)/2}z^{-3m};q^{3m^2}\right\rangle_\infty\Bigg],\end{aligned}$$

$$\beta_r=q^{r(3r+1)/2}\Bigg[z^{-3r}\left\langle -q^{m(3m+6r+1)}z^{-3m};q^{3m^2}\right\rangle_\infty$$

$$- z^{3r+1}\left\langle -q^{m(3m-6r-1)/2}z^{-3m};q^{3m^2}\right\rangle_\infty\Bigg].$$

If m is an odd positive integer, then

$$\langle z;q\rangle_\infty (qz^2,q/z^2;q^2)_\infty = \alpha_0 + \sum_{r=1}^{(m-1)/2}(\alpha_r+\beta_r), \tag{5.4}$$

while if m is an even positive integer, then

$$\langle z;q\rangle_\infty (qz^2,q/z^2;q^2)_\infty = \alpha_0 + \alpha_{m/2} + \sum_{r=1}^{m/2-1}(\alpha_r+\beta_r). \tag{5.5}$$

Proof. Replace q with $q^{3/2}$ and z with $-z^3/q^{1/2}$ in (1.9) to get that

$$\left\langle -qz^3;q^3\right\rangle_\infty = \sum_{r=0}^{m-1} f_r, \tag{5.6}$$

where

$$f_r := q^{r(3r-1)/2}z^{3r}\left\langle -q^{m(3m-6r+1)/2}z^{-3m};q^{3m^2}\right\rangle_\infty. \tag{5.7}$$

Likewise, replacing q with $q^{3/2}$ and z with $-z^3q^{1/2}$ in (1.9) gives

$$\left\langle -q/z^3;q^3\right\rangle_\infty = \sum_{r=0}^{m-1} g_r, \tag{5.8}$$

where

$$g_r := q^{r(3r+1)/2}z^{3r}\left\langle -q^{m(3m-6r-1)/2}z^{-3m};q^{3m^2}\right\rangle_\infty. \tag{5.9}$$

Note that (1.8) now implies that

$$\langle z;q\rangle_\infty (qz^2,q/z^2;q^2)_\infty = \sum_{r=0}^{m-1} f_r - z\sum_{r=0}^{m-1} g_r. \tag{5.10}$$

By using (1.11) and simplifying, it follows for $1\leq r\leq \lfloor (m-1)/2$, that

$$\begin{aligned} f_{m-r} &= q^{r(3r+1)/2}z^{-3r}\left\langle -q^{m(3m-6r-1)/2}z^{3m};q^{3m^2}\right\rangle_\infty \\ g_{m-r} &= q^{r(3r-1)/2}z^{-3r}\left\langle -q^{m(3m-6r+1)/2}z^{3m};q^{3m^2}\right\rangle_\infty \end{aligned}$$

Then for $1\leq r\leq \lfloor (m-1)/2\rfloor$,

$$\begin{aligned} f_r - zg_{m-r} &= \alpha_r, \\ f_{m-r} - zg_r &= \beta_r. \end{aligned}$$

Further, $f_0 - zg_0 = \alpha_0$, and if m is even, $f_{m/2} - zg_{m/2} = \alpha_{m/2}$, and the results now follows from (5.10) above. □

The following Lambert series identities are an easy consequence of Theorem 5.1.

Corollary 5.2. *Let $|q|<1$ and $m \geq 1$ be a positive integer. Define*
(5.11)

$$\begin{aligned}
\gamma_r &= q^{r(3r+1)/2} \left\langle -q^{m(3m-6r-1)/2}; q^{3m^2} \right\rangle_\infty \\
&\times \left[6r+1+6m\sum_{k=1}^{\infty} \frac{(-1)^k q^{km(3m-6r-1)/2}}{1-q^{3km^2}} - 6m \sum_{k=1}^{\infty} \frac{(-1)^k q^{km(3m+6r+1)/2}}{1-q^{3km^2}} \right], \\
\delta_r &= q^{r(3r-1)/2} \left\langle -q^{m(3m-6r+1)/2}; q^{3m^2} \right\rangle_\infty \\
&\times \left[6r-1+6m\sum_{k=1}^{\infty} \frac{(-1)^k q^{km(3m-6r+1)/2}}{1-q^{3km^2}} - 6m \sum_{k=1}^{\infty} \frac{(-1)^k q^{km(3m+6r-1)/2}}{1-q^{3km^2}} \right].
\end{aligned}$$

If m is an odd positive integer, then

$$(q;q)_\infty^3 (q;q^2)_\infty^2 = \gamma_0 + \sum_{r=1}^{(m-1)/2} (\gamma_r - \delta_r), \tag{5.12}$$

while if m is an even positive integer, then

$$(q;q)_\infty^3 (q;q^2)_\infty^2 = \gamma_0 - \delta_{m/2} + \sum_{r=1}^{m/2-1} (\gamma_r - \delta_r). \tag{5.13}$$

Proof. In each of the equations at (5.4) and (5.5), divide both sides by $1-z$ and let $z \to 1$. The remainder of the proof is similar to the proof of Theorem 4.1, and the details are omitted. $\square$

Remark: Note that (5.12) gives an m-dissection of $(q;q)_\infty^3(q;q^2)_\infty^2$ for the case of m odd, while (5.13) gives just an $m/2$ dissection in the case of m even.

As with Corollaries 4.1 and 4.2, it is possible to refine the results in Corollary 5.2 and derive expressions for some of the individual γ_r and δ_r in that corollary. The product $(q;q)_\infty^3(q;q^2)_\infty^2$ may be written as $(q;q)_\infty \left\langle q;q^2 \right\rangle_\infty^2$. Theorem 2.1 (in the case $m \geq 5$ is an integer relatively prime to 6) or the identities at (2.8) may be used to get an m-dissection of $(q;q)_\infty$ (an $m/2$ dissection in the case m is even). Likewise, Corollary 5.1 may be used to obtain an m-dissection of $\left\langle q;q^2 \right\rangle_\infty$, and hence by multiplying these expansions together, an m-dissection (or $m/2$ dissection when m is even) of $(q;q)_\infty \left\langle q;q^2 \right\rangle_\infty^2$ may be obtained. Thus if this expansion (in the case m is odd) is denoted by

$$(q;q)_\infty \left\langle q;q^2 \right\rangle_\infty^2 = \sum_{r=0}^{m-1} R_{m,r}(q^m) q^r,$$

and the right (Lambert series) side of (5.12) is denoted by

$$(q;q)_\infty^3 (q;q^2)_\infty^2 = \sum_{r=0}^{m-1} L_{m,r}(q^m) q^r,$$

identities involving individual γ_r and δ_r in Corollary 5.2 may be obtained by equating $R_{m,r}(q^m)$ and $L_{m,r}(q^m)$, and replacing q^m with q. A similar statement holds in the case where m is even, except that as noted above, the dissections involved are $m/2$ - dissections.

Before stating some examples in the next corollary, we note that for the most part these identities are not particulary elegant, as on the one hand the $R_{m,r}$ usually involve sums of several infinite products, and the $L_{m,r}$ may involve more than one of the γ_r and/or δ_r.

Corollary 5.3. *Let* $|q| < 1$.

$$1+6\sum_{k=1}^{\infty}\frac{(-1)^k q^k}{1-q^{3k}}-6\sum_{k=1}^{\infty}\frac{(-1)^k q^{2k}}{1-q^{3k}}=\frac{f_1^6 f_6}{f_2^3 f_3^2}. \tag{5.14}$$

$$\begin{aligned}1+18\sum_{k=1}^{\infty}\frac{(-1)^k q^{4k}}{1-q^{9k}}&-18\sum_{k=1}^{\infty}\frac{(-1)^k q^{5k}}{1-q^{9k}}\\ =\frac{1}{\langle -q^4,q^9\rangle_\infty}\Big[&\langle q^4,q^9\rangle_\infty\langle q^3,q^6\rangle_\infty^2\\ &+4q\langle q,q^6\rangle_\infty\left[\langle q,q^9\rangle_\infty\langle q^3,q^6\rangle_\infty-\langle q,q^6\rangle_\infty\langle q^2,q^9\rangle_\infty\right]\Big].\end{aligned} \tag{5.15}$$

$$\begin{aligned}5+30\sum_{k=1}^{\infty}\frac{(-1)^k q^{5k}}{1-q^{15k}}&-30\sum_{k=1}^{\infty}\frac{(-1)^k q^{10k}}{1-q^{15k}}=\frac{f_5}{\langle -q^5,q^{15}\rangle_\infty}\\ \times\Bigg[&\langle q^5,q^{10}\rangle_\infty^2+\frac{4\langle q^2,q^5\rangle_\infty\langle q^3,q^{10}\rangle_\infty\langle q^5,q^{10}\rangle_\infty}{\langle q,q^5\rangle_\infty}\\ &+\frac{4q\langle q,q^5\rangle_\infty\langle q,q^{10}\rangle_\infty\langle q^5,q^{10}\rangle_\infty}{\langle q^2,q^5\rangle_\infty}-8q\langle q,q^{10}\rangle_\infty\langle q^3,q^{10}\rangle_\infty\Bigg]\end{aligned} \tag{5.16}$$

Proof. With the notation defined above, the identities in the corollary follow from comparing $R_{m,r}(q^m)$ with $L_{m,r}(q^m)$ with (m,r) being respectively, $(1,0)$, $(3,0)$ and $(5,1)$ (with Theorem 2.1 being used for the 5-dissection of $(q;q)_\infty$). □

The anonymous referee pointed out that the left-hand side of equation (5.16) is 5 times the left-hand side of equation (5.14) with q replaced by q^5. This is not an isolated phenomenon. Indeed, if $m \equiv 5 \pmod 6$, then

$$\delta_{(m+1)/6}=$$
$$mq^{(m^2-1)/24}\left\langle -q^{m^2};q^{3m^2}\right\rangle_\infty\left(1+6\sum_{k=1}^{\infty}\frac{(-1)^k q^{km^2}}{1-q^{3km^2}}-6\sum_{k=1}^{\infty}\frac{(-1)^k q^{2km^2}}{1-q^{3km^2}}\right),$$

and if $m \equiv 1 \pmod 6$, then $\gamma_{(m-1)/6}$ is equal to the same expression. In either case, the combination of Lambert series inside the ( ) reduces to the left side of (5.14) after making the substitution $q^{m^2} \to q$.

Remark: The author initially thought that the Ramanujan-like Lambert series identity at (5.14) was new, but the anonymous referee also pointed out that it follows fairly trivially from an identity in [3, page 229] (by expanding the denominators of the Lambert series there as series, and then changing the order of summation).

We next consider when $4|m$ and give an example of using (5.13) (which gives an $m/2$-dissection of $(q;q)_\infty^3(q;q^2)_\infty^2$) in conjunction with the $m/2$ case of (5.2) (which gives an $m/2$-dissection of $\langle q;q^2\rangle_\infty$) and the m case of (2.8) (which gives an $m/2$-dissection of $(q;q)_\infty$) to produce identities. As in the previous corollary, the identities come from equating the expansion coming from $(q;q)_\infty \times \langle q;q^2\rangle_\infty^2$ (using (2.8) and (5.2)) with that coming from $(q;q)_\infty^3(q;q^2)_\infty^2$ (using (5.13)), setting equal components in each expansion containing powers of q in the same arithmetic progression $\pmod{m/2}$, and finally making the replacement $q^{m/2} \to q^{m/4}$.

Corollary 5.4. *If* $|q| < 1$, *then*

$$\begin{aligned}(5.17)\quad &\langle -q^{11};q^{24}\rangle_\infty \left(1+24\sum_{k=1}^{\infty}\frac{(-1)^k q^{11k}}{1-q^{24k}} - 24\sum_{k=1}^{\infty}\frac{(-1)^k q^{13k}}{1-q^{24k}}\right)\\
&+q\,\langle -q^5;;q^{24}\rangle_\infty \left(7+24\sum_{k=1}^{\infty}\frac{(-1)^k q^{5k}}{1-q^{24k}} - 24\sum_{k=1}^{\infty}\frac{(-1)^k q^{19k}}{1-q^{24k}}\right)\\
&=\left[\langle -q^2;q^4\rangle_\infty^2 + q\,\langle -1;q^4\rangle_\infty^2\right]\left[\langle -q^{11};q^{24}\rangle_\infty - q\,\langle -q^5;q^{24}\rangle_\infty\right]\\
&\quad + 2q\,\langle -1;q^4\rangle_\infty \langle -q^2;q^4\rangle_\infty \left[\langle -q^7;q^{24}\rangle_\infty - q^2\,\langle -q;q^{24}\rangle_\infty\right];\end{aligned}$$

$$\begin{aligned}(5.18)\quad &\langle -q^{7};q^{24}\rangle_\infty \left(5+24\sum_{k=1}^{\infty}\frac{(-1)^k q^{7k}}{1-q^{24k}} - 24\sum_{k=1}^{\infty}\frac{(-1)^k q^{17k}}{1-q^{24k}}\right)\\
&+q^2\,\langle -q;;q^{24}\rangle_\infty \left(11+24\sum_{k=1}^{\infty}\frac{(-1)^k q^{k}}{1-q^{24k}} - 24\sum_{k=1}^{\infty}\frac{(-1)^k q^{23k}}{1-q^{24k}}\right)\\
&=\left[\langle -q^2;q^4\rangle_\infty^2 + q\,\langle -1;q^4\rangle_\infty^2\right]\left[\langle -q^{7};q^{24}\rangle_\infty - q^2\,\langle -q;q^{24}\rangle_\infty\right]\\
&\quad + 2\,\langle -1;q^4\rangle_\infty \langle -q^2;q^4\rangle_\infty \left[\langle -q^{11};q^{24}\rangle_\infty - q\,\langle -q^5;q^{24}\rangle_\infty\right].\end{aligned}$$

Proof. In line with the discussion immediately preceding this corollary, these identities follow from setting $m = 4$ in (5.13) (which gives a 2-dissection of $(q;q)_\infty^3(q;q^2)_\infty^2$), $m = 2$ in (5.2) (which gives a 2-dissection of $\langle q;q^2\rangle_\infty$) and the $m = 4$ in (2.8) (which gives a 2-dissection of $(q;q)_\infty$). The first identity comes from equating the even part of each expansion (that of $(q;q)_\infty \times \langle q;q^2\rangle_\infty^2$ and that of $(q;q)_\infty^3(q;q^2)_\infty^2$), while the second identity comes from equating the odd parts, after making the replacement $q^2 \to q$ in each case. $\square$

There is another expression for the infinite product on the left sides of (5.12) and (5.13), which is stated by Fine [8, p.83]:

$$\sum_{n=-\infty}^{\infty} (6n+1)q^{n(3n+1)/2} = (q;q)_\infty^3 (q;q^2)_\infty^2. \tag{5.19}$$

We had initially hoped to use the m-dissection of this identity in conjunction with the m-dissection of (5.13) to produce new identities with left sides similar to those in Corollaries 4.3 and 4.4. Indeed it is possible to produce identities such as the following:

$$\begin{aligned}\sum_{k=-\infty}^{\infty} &\left((30k+7)q^{k(15k+7)/2} + (30k+13)q^{\left(15k^2+13k\right)/2+1}\right) \\ &= \left\langle -q^4;q^{15}\right\rangle_\infty \left(7 + 30\sum_{k=1}^{\infty} \frac{(-1)^k q^{4k}}{1-q^{15k}} - 30\sum_{k=1}^{\infty} \frac{(-1)^k q^{11k}}{1-q^{15k}}\right) \\ &\quad + q\left\langle -q;q^{15}\right\rangle_\infty \left(13 + 30\sum_{k=1}^{\infty} \frac{(-1)^k q^{k}}{1-q^{15k}} - 30\sum_{k=1}^{\infty} \frac{(-1)^k q^{14k}}{1-q^{15k}}\right).\end{aligned} \tag{5.20}$$

Unfortunately, it appears that any attempt to compare m - dissections coming from (5.13) with those coming from (5.19) will lead to identities that follow directly from a combined use of the Jacobi triple product identity

$$\sum_{n=-\infty}^{\infty} z^n q^{n(n-1)/2} = \langle -z;q\rangle_\infty,$$

with the identity that follows from differentiating both sides of the Jacobi triple product identity with respect to z (valid for $|q| < |z| < 1$):

$$\sum_{n=-\infty}^{\infty} nz^n q^{n(n-1)/2} = \langle -z;q\rangle_\infty \left(\sum_{k=1}^{\infty} \frac{(-1)^k \left(\frac{q}{z}\right)^k}{1-q^k} - \sum_{k=1}^{\infty} \frac{(-1)^k z^k}{1-q^k}\right). \tag{5.21}$$

It can then be easily seen that the sum over the first term on the left of (5.20) is equal to the first term on the right side ($q \to q^{15}$, $z \to q^{11}$), and that the sum over the second term on the left of (5.20) is equal to the second term on the right side ($q \to q^{15}$, $z \to q^{14}$).

However, it is possible to derive identities that do not follow so trivially by comparing the m-dissection of (5.19) with the m-dissection of the product $[(q;q)_\infty][\left\langle q;q^2\right\rangle_\infty^2]$, as was done to get the right sides in Corollary 5.3. For example, taking the 3-dissection of each, then comparing the components containing powers of q in the arithmetic progression 0 (mod 3) and finally making the replacement $q^3 \to q$, leads to the identity in the next corollary.

Corollary 5.5. *If $|q| < 1$, then*

$$\sum_{k=-\infty}^{\infty}(18k+1)q^{(9k^2+k)/2} = \langle q^4, q^9\rangle_\infty \langle q^3, q^6\rangle_\infty^2 + 4q\langle q, q^6\rangle_\infty \left[\langle q, q^9\rangle_\infty \langle q^3, q^6\rangle_\infty - \langle q, q^6\rangle_\infty \langle q^2, q^9\rangle_\infty\right]. \tag{5.22}$$

6. Concluding Remarks

In Theorems 2.1, 2.2, 3.1 and 3.2 in the present paper, $m \geq 5$ was an integer relatively prime to 6. An obvious question to ask is if anything can be said in the case where m lies in some other class of positive integers.

As was seen in these theorems, the key to combining pairs of triple products to form quintuple products was patterns in various lists of residues modulo m. In Theorem 2.1, for example, this meant the existence of two symmetric subsequences in the list

$$\left\{\frac{r(3r-1)}{2} : r = 0, 1, 2, \ldots m-1\right\}. \tag{6.1}$$

It is easy to give a family of values of m, namely m of the form $m = 3^k$ for which the set of numbers in (6.1) are all distinct modulo m, so that it is impossible to combine pairs of terms in (2.1) to get quintuple products.

When m has the form $m = 2^k$, then the list (6.1) consists of a palindromic part and an unmatched part, as is illustrated by $m = 16$:

$$\underbrace{0, 1, 5, 12, 6, 3, 3, 6, 12, 5, 1, 0}, 2, 7, 15, 10.$$

In such a situation it may be possible to express $(q; q)_\infty$ as sum containing quintuple products and triple products.

The situation may be less complicated for Theorem 3.1 in the case m is even and relatively prime to 3 (since the relevant congruences, $3r, 3r + 1 \pmod{m}$, are linear), and results similar to those in Theorem 3.1 may be obtainable.

We leave it to the interested reader to see what they can discover.

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