A CONVERGENCE THEOREM FOR CONTINUED FRACTIONS OF THE FORM $K_{n=1}^{\infty} a_n/1$.

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ABSTRACT. In this paper we present a convergence theorem for continued fractions of the form $K_{n=1}^{\infty} a_n/1$. By deriving conditions on the a_n which ensure that the odd and even parts of $K_{n=1}^{\infty} a_n/1$ converge, these same conditions also ensure that they converge to the same limit. Examples will be given.

This paper is dedicated to Professor Olav Njastad on the occasion of his 70th birthday.

1. INTRODUCTION

In this paper we derive a convergence theorem for continued fractions of the form $K_{n=1}^{\infty} a_n/1$. This is achieved by using Worpitzky's theorem to derive conditions on the a_n which ensure that the odd and even parts of $K_{n=1}^{\infty} a_n/1$ converge and, furthermore, converge to the same limit. We will assume $a_n \neq 0$ for any n, since otherwise the continued fraction is finite and converges trivially in $\hat{\mathbb{C}}$.

We begin by summarizing definitions and basic properties for continued fractions that are needed. We write A_N/B_N (the N-th *approximant*) for the finite continued fraction $b_0 + K_{n=1}^{N} a_n/b_n$ written as a rational function of the variables $a_1, \ldots, a_N, b_1, \ldots, b_N$. It is elementary that the A_N (the N-th (canonical) *numerator*) and B_N (the N-th (canonical) *denominator*) satisfy the following recurrence relations:

$$\begin{aligned} A_N &= b_N A_{N-1} + a_N A_{N-2}, A_{-1} = 1, A_0 = 0, \\ B_N &= b_N B_{N-1} + a_N B_{N-2}, B_{-1} = 0, B_0 = 1. \end{aligned} \tag{1.1}$$

It can also be easily shown that

$$A_N B_{N-1} - A_{N-1} B_N = (-1)^{N-1} \prod_{i=1}^{N} a_i. \tag{1.2}$$

We call $d_0 + K_{n=1}^{\infty} c_n/d_n$ a *canonical contraction* of $b_0 + K_{n=1}^{\infty} a_n/b_n$ if

$$C_k = A_{n_k}, \qquad D_k = B_{n_k} \qquad \text{for } k = 0, 1, 2, 3, \ldots,$$

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where C_n, D_n, A_n and B_n are canonical numerators and denominators of $d_0 + K_{n=1}^{\infty} c_n/d_n$ and $b_0 + K_{n=1}^{\infty} a_n/b_n$ respectively.

From [5] (page 83) we have the following theorem:

Theorem 1. *The canonical contraction of $b_0 + K_{n=1}^{\infty} a_n/b_n$ with*

$$C_k = A_{2k} \qquad D_k = B_{2k} \qquad \text{for } k = 0, 1, 2, 3, \dots ,$$

exists if and only if $b_{2k} \neq 0 for k = 0, 1, 2, 3, \dots$, and in this case is given by

$$b_0 + \frac{b_2 a_1}{b_2 b_1 + a_2} - \frac{a_2 a_3 b_4/b_2}{a_4 + b_3 b_4 + a_3 b_4/b_2} - \frac{a_4 a_5 b_6/b_4}{a_6 + b_5 b_6 + a_5 b_6/b_4} - \cdots . \tag{1.3}$$

The continued fraction (1.3) is called the *even* part of $b_0 + K_{n=1}^{\infty} a_n/b_n$.

From [5] (page 85) we also have:

Theorem 2. *The canonical contraction of $b_0 + K_{n=1}^{\infty} a_n/b_n$ with $C_0 = A_1/B_1$, $D_0 = 1$ and*

$$C_k = A_{2k+1} \qquad D_k = B_{2k+1} \qquad \text{for } k = 1, 2, 3, \dots ,$$

exists if and only if $b_{2k+1} \neq 0$ for $k = 0, 1, 2, 3, \dots$, and in this case is given by

$$\begin{aligned} &\frac{b_0 b_1 + a_1}{b_1} - \frac{a_1 a_2 b_3/b_1}{b_1(a_3 + b_2 b_3) + a_2 b_3} - \frac{a_3 a_4 b_5 b_1/b_3}{a_5 + b_4 b_5 + a_4 b_5/b_3} \\ &\qquad - \frac{a_5 a_6 b_7/b_5}{a_7 + b_6 b_7 + a_6 b_7/b_5} - \frac{a_7 a_8 b_9/b_7}{a_9 + b_8 b_9 + a_8 b_9/b_7} - \cdots . \end{aligned} \tag{1.4}$$

The continued fraction (1.4) is called the *odd* part of $b_0 + K_{n=1}^{\infty} a_n/b_n$.

We also make repeated use of

Worpitzky's Theorem ([5], *pp. 35–36*) *For all $n \geq 1$, let*

$$|a_n| \leq \frac{1}{4}.$$

Then $K_{n=1}^{\infty} a_n/1$ converges. All approximants f_n of the continued fraction lie in the disc $|w| < 1/2$ and the value of the continued fraction f is in the disk $|w| \leq 1/2$.

2. Background

Of course the idea of using the convergence of the odd and even parts of a continued fraction $K_{n=1}^{\infty} a_n/1$ to show that the continued fraction itself converges is not new. The following system of inequalities, called the *fundamental inequalities* by Wall [8], ensures the convergence of the odd and even parts of $K_{n=1}^{\infty} a_n/1$.

$$\begin{aligned} r_1|1 + a_1| &\geq |a_1|, \\ r_2|1 + a_1 + a_2| &\geq |a_2|, \\ r_n|1 + a_{n-1} + a_n| &\geq r_n r_{n-2}|a_{n-1}| + |a_n|, \quad n = 3, 4, 5, \dots , \end{aligned} \tag{2.1}$$

where $r_n \geq 0$. These inequalities are obtained by applying the Śleszyński-Pringsheim criterion stated below to continued fractions equivalent to the even and odd parts of $K_{n=1}^{\infty} a_n/1$. This approach has been the starting point for several important lines of research (see [8], Chapter III and [2], section 4.4.5, for more details of these).

Śleszyński-Pringsheim Theorem ([6], *see also* [2], *page 92*) *The continued fraction* $K(a_n/b_n)$ *converges to a finite value if*

$$|b_n| \geq |a_n| + 1, \qquad n = 1, 2, 3, \ldots. \tag{2.2}$$

If f_n *denotes the n-th approximant, then*

$$|f_n| < 1, \qquad n = 1, 2, 3, \ldots. \tag{2.3}$$

Rather than working with very general implicit inequalities, such as those at (2.1), we have looked for simple explicit inequalities. This approach allows us to state some quite general, simple criteria for the convergence of certain classes of continued fractions of the form $K_{n=1}^{\infty} a_n/1$. We have the following theorem.

3. Main Theorem and Examples

Theorem 3. *Let* $\{c_n\}_{n=1}^{\infty}$ *and* $\{\beta_n\}_{n=1}^{\infty}$ *be sequences of numbers, with* $c_n > 0$ *and* $0 < \beta_n < 1$ *for* $n \geq 1$. *If the terms in the sequence* $\{a_n\}_{n=1}^{\infty}$ *satisfy, for* $n \geq 1$, *either*

$$(3.1) \qquad |a_{2n}| \geq \frac{c_n + 1}{1 - \beta_n},$$

$$|a_{2n+1}| \leq \min\left\{c_n,\, c_{n+1},\, \beta_n\beta_{n+1}\frac{c_{n+1} + 1}{4(1 - \beta_{n+1})},\, \beta_n\beta_{n+1}\frac{c_n + 1}{4(1 - \beta_n)}\right\},$$

or

$$(3.2) \qquad |a_{2n}| \geq \max\left\{\frac{c_n + 1}{1 - \beta_n}, \frac{c_{n+1} + 1}{1 - \beta_{n+1}}\right\},$$

$$|a_{2n+1}| \leq \min\left\{c_n,\, c_{n+1},\, \beta_n\beta_{n+1}\frac{c_{n+1} + 1}{4(1 - \beta_{n+1})}\right\},$$

then the continued fraction $K_{n=1}^{\infty} a_n/1$ *converges to a finite value.*

Proof. By Theorem 1 the even part of $K_{n=1}^{\infty} a_n/1$ is

(3.3)

$$\frac{a_1}{1 + a_2} - \frac{a_2 a_3}{a_4 + 1 + a_3} - \frac{a_4 a_5}{a_6 + 1 + a_5} - \cdots - \frac{a_{2n} a_{2n+1}}{a_{2n+2} + 1 + a_{2n+1}} - \cdots$$

$$= \frac{a_1/(1 + a_2)}{1} - \frac{a_2 a_3/((1 + a_2)(a_4 + 1 + a_3))}{1}$$
$$- \frac{a_4 a_5/((a_4 + 1 + a_3)(a_6 + 1 + a_5))}{1} - \cdots$$

$$-\frac{a_{2n}a_{2n+1}/((a_{2n}+1+a_{2n-1})(a_{2n+2}+1+a_{2n+1}))}{1}-\cdots.$$

The second continued fraction arises from the first after applying a sequence of similarity transformations. We now show that a tail of this continued fraction satisfies the conditions of Worpitzky's Theorem. From the conditions at (3.1) or (3.2) above, it follows that $|a_{2i}| > c_i + 1 \geq |a_{2i-1}| + 1$ and so, using the conditions at (3.1) or (3.2) several times, we have that

$$\begin{aligned}
&\left|\frac{a_{2n}a_{2n+1}}{(a_{2n}+1+a_{2n-1})(a_{2n+2}+1+a_{2n+1})}\right| \\
&\leq \frac{|a_{2n}||a_{2n+1}|}{(|a_{2n}|-(c_n+1))(|a_{2n+2}|-(c_{n+1}+1))} \\
&\leq \frac{|a_{2n}||a_{2n+1}|}{(|a_{2n}|-(1-\beta_n)|a_{2n}|)(|a_{2n+2}|-(1-\beta_{n+1})|a_{2n+2}|)} \\
&= \frac{|a_{2n+1}|}{\beta_n\beta_{n+1}|a_{2n+2}|} \leq \frac{|a_{2n+1}|}{\beta_n\beta_{n+1}(c_{n+1}+1)/(1-\beta_{n+1})} \leq \frac{1}{4}.
\end{aligned}$$

Thus, by Worpitzky's Theorem, the even part of $K_{n=1}^{\infty} a_n/1$ equals

$$\frac{a_1/(1+a_2)}{1} - \frac{a_2a_3/((1+a_2)(a_4+1+a_3))}{1+\alpha},$$

for some α with $|\alpha| \leq 1/2$. Likewise,

$$\begin{aligned}
\left|\frac{a_2a_3}{(1+a_2)(a_4+1+a_3)}\right| &\leq \frac{|a_3|}{(1-1/|a_2|)\beta_2|a_4|} \\
&\leq \frac{\beta_1\beta_2(c_2+1)/(4(1-\beta_2))}{\left(1-\dfrac{1-\beta_1}{c_1+1}\right)\beta_2\dfrac{c_2+1}{1-\beta_2}} \\
&= \frac{\beta_1(c_1+1)}{4(c_1+\beta_1)} < \frac{1}{2}.
\end{aligned}$$

Hence the even part of $K_{n=1}^{\infty} a_n/1$ converges to a finite value.

From Theorem 2, the odd part of $K_{n=1}^{\infty} a_n/1$ is

(3.4)

$$\frac{a_1}{1} - \frac{a_1a_2}{a_3+1+a_2} - \frac{a_3a_4}{a_5+1+a_4} - \frac{a_5a_6}{a_7+1+a_6} - \cdots \frac{a_{2n-1}a_{2n}}{a_{2n+1}+1+a_{2n}} - \cdots$$

$$\sim \frac{a_1}{1} - \frac{a_1a_2/(a_3+1+a_2)}{1} - \frac{a_3a_4/((a_3+1+a_2)(a_5+1+a_4))}{1} - \cdots$$
$$-\frac{a_{2n-1}a_{2n}/((a_{2n-1}+1+a_{2n-2})(a_{2n+1}+1+a_{2n}))}{1}-\cdots.$$

By similar reasoning to that used above,

$$\left|\frac{a_{2n-1}a_{2n}}{(a_{2n-1}+1+a_{2n-2})(a_{2n+1}+1+a_{2n})}\right| \leq \frac{|a_{2n-1}|}{\beta_n\beta_{n-1}|a_{2n-2}|} \leq \frac{1}{4}.$$

Thus, again by Worpitzky's theorem, the odd part of $K_{n=1}^{\infty} a_n/1$ equals

$$\frac{a_1}{1} - \frac{a_1 a_2/(a_3+1+a_2)}{1+\alpha'},$$

for some α' with $|\alpha'| \leq 1/2$. Thus the odd part of $K_{n=1}^{\infty} a_n/1$ converges to a finite value provided $|a_3 + 1 + a_2| > 0$, and this follows easily from the inequalities satisfied by $|a_2|$ and $|a_3|$ in the statement of the theorem.

We next show that the conditions at (3.1) and (3.2) are also sufficient to show that the odd and even parts tend to the same limits. From the recurrence relations at (1.1) and Equation 1.2, it follows that

$$\left|\frac{A_{2n+1}}{B_{2n+1}} - \frac{A_{2n-1}}{B_{2n-1}}\right| = \frac{\prod_{i=1}^{2n} |a_i|}{|B_{2n+1}B_{2n-1}|}. \tag{3.5}$$

Since the sequence $\{A_{2i+1}/B_{2i+1}\}$ converges to a finite value, it follows that the expression on the right tends to 0 as $n \to \infty$. Next,

$$\left|\frac{A_{2n}}{B_{2n}} - \frac{A_{2n-1}}{B_{2n-1}}\right| = \frac{\prod_{i=1}^{2n} |a_i|}{|B_{2n}B_{2n-1}|} = \frac{\prod_{i=1}^{2n} |a_i|}{|B_{2n+1}B_{2n-1}|}\frac{|B_{2n+1}|}{|B_{2n}|}. \tag{3.6}$$

Thus, if it can be shown that the sequence $\{B_{2n+1}/B_{2n}\}$ is bounded, then the left side of (3.6) tends to 0, the odd and even parts of $K_{n=1}^{\infty} a_n/1$ tend to the same limits and the continued fraction converges to a finite value.

From the second equation at (1.1) and Theorem 1 we have that

$$\frac{B_{2n+1}}{B_{2n}} = 1 + \frac{a_{2n+1}}{1} + \frac{a_{2n}}{1} + \frac{a_{2n-1}}{1} + \cdots + \frac{a_2}{1}$$

$$= 1 + \frac{a_{2n+1}}{1+a_{2n}} - \frac{a_{2n}a_{2n-1}}{a_{2n-2}+1+a_{2n-1}} - \frac{a_{2n-2}a_{2n-3}}{a_{2n-4}+1+a_{2n-3}} - \cdots - \frac{a_{2n-2j}a_{2n-2j-1}}{a_{2n-2j-2}+1+a_{2n-2j-1}} - \cdots - \frac{a_4a_3}{a_2+1+a_3}$$

$$= 1 + \frac{a_{2n+1}/(1+a_{2n})}{1} - \frac{a_{2n}a_{2n-1}/((1+a_{2n})(a_{2n-2}+1+a_{2n-1}))}{1} - \frac{a_{2n-2}a_{2n-3}/((a_{2n-2}+1+a_{2n-1})(a_{2n-4}+1+a_{2n-3}))}{1} - \cdots - \frac{a_{2n-2j}a_{2n-2j-1}/((a_{2n-2j}+1+a_{2n-2j+1})(a_{2n-2j-2}+1+a_{2n-2j-1}))}{1} - \cdots - \frac{a_4a_3/((a_4+1+a_5)(a_2+1+a_3))}{1}.$$

These equalities are valid since the given continued fraction expansion of B_{2n+1}/B_{2n} is finite. Once again using the conditions at (3.1) or (3.2), we have that

$$\left|\frac{a_{2n-2j}a_{2n-2j-1}}{(a_{2n-2j}+1+a_{2n-2j+1})(a_{2n-2j-2}+1+a_{2n-2j-1})}\right|$$

$$\leq \frac{|a_{2n-2j}||a_{2n-2j-1}|}{(|a_{2n-2j}|-(c_{n-j}+1))(|a_{2n-2j-2}|-(c_{n-j-1}+1))}$$
$$\leq \frac{|a_{2n-2j}||a_{2n-2j-1}|}{(|a_{2n-2j}|-(1-\beta_{n-j})|a_{2n-2j}|)(|a_{2n-2j-2}|-(1-\beta_{n-j-1})|a_{2n-2j-2}|)}$$
$$= \frac{|a_{2n-2j-1}|}{\beta_{n-j}\beta_{n-j-1}|a_{2n-2j-2}|} \leq \frac{|a_{2n-2j-1}|}{\beta_{n-j}\beta_{n-j-1}(c_{n-j-1}+1)/(1-\beta_{n-j-1})} \leq \frac{1}{4}.$$

Similarly,

$$\begin{aligned}\left|\frac{a_{2n}a_{2n-1}}{(1+a_{2n})(a_{2n-2}+1+a_{2n-1})}\right| &\leq \frac{|a_{2n-1}|}{(1-1/|a_{2n}|)\beta_{n-1}|a_{2n-2}|}\\ &\leq \frac{\beta_n\beta_{n-1}(c_{n-1}+1)/(4(1-\beta_{n-1}))}{\frac{c_n+\beta_n}{c_n+1}\beta_{n-1}\frac{c_{n-1}+1}{1-\beta_{n-1}}}\\ &= \frac{\beta_n(c_n+1)}{4(c_n+\beta_n)} \leq \frac{1}{4}\end{aligned}$$

Thus, by Worpitzky's theorem, there exists α with $|\alpha| \leq 1/2$ such that

$$\left|\frac{B_{2n+1}}{B_{2n}}\right| = \left|1+\frac{a_{2n+1}/(1+a_{2n})}{1+\alpha}\right| \leq 1+2\frac{|a_{2n+1}|}{|a_{2n}|-1} \leq 3.$$

Thus the sequence $\{B_{2n+1}/B_{2n}\}$ is bounded by 3 and $K_{n=1}^{\infty} a_n/1$ converges to a finite value. □

If $\{c_n\}$ and $\{\beta_n\}$ are increasing sequences, the statement of the theorem is simplified a little and we get the following corollary.

Corollary 1. *Let $\{c_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be increasing sequences of numbers, with $c_n > 0$ and $0 < \beta_n < 1$ for $n \geq 1$. If the terms in the sequence $\{a_n\}_{n=1}^{\infty}$ satisfy, for $n \geq 1$, either*

$$|a_{2n}| \geq \frac{c_n+1}{1-\beta_n}, \qquad |a_{2n+1}| \leq \min\left\{c_n,\, \beta_n\beta_{n+1}\frac{c_n+1}{4(1-\beta_n)}\right\},$$

or

$$|a_{2n}| \geq \frac{c_{n+1}+1}{1-\beta_{n+1}}, \qquad |a_{2n+1}| \leq \min\left\{c_n,\, \beta_n\beta_{n+1}\frac{c_{n+1}+1}{4(1-\beta_{n+1})}\right\},$$

then the continued fraction $K_{n=1}^{\infty} a_n/1$ converges to a finite value.

In Corollary 1 the numbers $(c_n+1)/(1-\beta_n)$, c_n and $\beta_n\beta_{n+1}(c_n+1)/(4(1-\beta_n))$ increase with n and so the corollary gives a convergence criterion for continued fractions $K_{n=1}^{\infty} a_n/1$ in which both the even- and odd-indexed partial numerators can become arbitrarily large. The necessity to find the minimum of c_n and $\beta_n\beta_{n+1}(c_n+1)/(4(1-\beta_n))$ or of c_n and $\beta_n\beta_{n+1}(c_{n+1}+1)/(4(1-\beta_{n+1}))$ is a little cumbersome. We also have the following corollaries which give cleaner conditions on the a_n.

It is also of interest to be able to prove the convergence of continued fractions where both the odd and even-indexed partial numerators become unbounded. Many convergence theorems require that infinitely many of the a_n lie inside some fixed bounded disc for the continued fraction $K_{n=1}^{\infty} a_n/1$ to converge. Hayden's theorem [1] (see also [2], page 126) requires at least one of a_n, a_{n+1} to lie inside the unit disc, for each $n \geq 1$. For $K_{n=1}^{\infty} c_n^2/1$ to converge, Lange's theorem [3] (see also [2], page 124) requires $|c_{2n-1} \pm i\, a| \leq \rho$, where a is a complex number and ρ is real number satisfying

$$|a| < \rho < |a+1|.$$

Our theorem allows us to prove the convergence of certain continued fractions $K_{n=1}^{\infty} a_n/1$, where the sequence $\{a_n\}$ does not contain any bounded subsequence.

Corollary 2. *Let $\{d_n\}_{n=1}^{\infty}$ be an increasing sequence of positive numbers, with $d_n > 25$ for $n \geq 1$. Suppose the terms of the sequence $\{a_n\}$ satisfy*

$$|a_{2n}| \geq d_n,$$
$$|a_{2n+1}| \leq \frac{4}{25}\, d_n$$

Then the continued fraction $K_{n=1}^{\infty} a_n/1$ converges to a finite value.

Proof. For $n \geq 1$, let $c_n = d_n/5 - 1$ and $\beta_n = 4/5$ in Theorem 3 and use the conditions at (3.1). □

Example 1. *Let the terms of the sequence $\{a_n\}$ satisfy*

$$|a_{2n-1}| = 4n, \qquad |a_{2n}| = 25n.$$

Then $K_{n=1}^{\infty} a_n/1$ converges.

Corollary 3. *Let $c > 0$. Suppose the terms in the sequence $\{a_n\}$ satisfy*

$$|a_{2n}| \geq 1 + 3c + 2\sqrt{c}\sqrt{2c+1},$$
$$|a_{2n+1}| \leq c.$$

Then the continued fraction $K_{n=1}^{\infty} a_n/1$ converges to a finite value.

Proof. In Theorem 3, let $c_n = c$ and $\beta_n = 2(\sqrt{c}\sqrt{2c+1} - c)/(c+1)$, for $n = 1, 2, 3, \ldots$. □

For large c this is clearly a weaker result than that of Thron [7], which states the following (see [2], page 124):

For $\rho > 1$, $K_{n=1}^{\infty} a_n/1$ converges to a finite value provided that

$$|a_{2n-1}| \leq \rho^2, \qquad |a_{2n}| \geq 2(\rho^2 - \cos\arg a_{2n}), \qquad n = 1, 2, 3, \ldots .$$

However, for small c it is possible to prove the convergence of certain continued fractions whose convergence cannot be proved by Thron's result. We have the following example.

Example 2. *Let the terms of the sequence* $\{a_n\}$ *satisfy*

$$|a_{2n}| \geq \frac{36}{23}, \qquad |a_{2n+1}| \leq \frac{1}{23},$$

with $a_{2k} = -36/23$ *for infinitely many* k. *Then the continued fraction* $K_{n=1}^{\infty} a_n/1$ *converges to a finite value.*

This follows from Corollary 3 with $c = 1/23$. Thron's theorem does not give the convergence of the continued fraction in this example, since there is no real $\rho > 1$ satisfying $36/23 \geq 2(\rho^2 + 1)$ (Hayden's theorem [1] also gives the convergence of this continued fraction).

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