POLYNOMIAL SOLUTIONS TO PELL'S EQUATION AND FUNDAMENTAL UNITS IN REAL QUADRATIC FIELDS

J. MC LAUGHLIN

Abstract. Finding polynomial solutions to Pell's equation is of interest as such solutions sometimes allow the fundamental units to be determined in an infinite class of real quadratic fields.

In this paper, for each triple of positive integers (c, h, f) satisfying

$$c^2 - f\,h^2 = 1,$$

where (c, h) are the smallest pair of integers satisfying this equation, several sets of polynomials $(c(t), h(t), f(t))$ which satisfy

$$c(t)^2 - f(t)\,h(t)^2 = 1 \text{ and } (c(0), h(0), f(0)) = (c, h, f)$$

are derived. Moreover, it is shown that the pair $(c(t), h(t))$ constitute the fundamental polynomial solution to the Pell's equation above.

The continued fraction expansion of $\sqrt{f(t)}$ is given in certain general cases (for example, when the continued fraction expansion of $\sqrt{f}$ has odd period length, or even period length or has period length $\equiv 2 \mod 4$ and the middle quotient has a particular form etc).

Some applications to determining the fundamental unit in real quadratic fields is also discussed.

1. Introduction: Polynomial Solution's to Pell's Equation

Finding polynomial solutions to Pell's equation is of interest as such solutions sometimes allow the fundamental units to be determined in an infinite class of real quadratic fields. Finding such polynomial solutions is not only an interesting problem in its own right, but knowing the fundamental unit in a real quadratic field means that the class number of the field can be found via some version of Dirichlet's class number formula (see, for example, [12]).

In [10], Perron gives some examples of polynomials f for which the continued fraction expansion of $\sqrt{f}$ can be stated explicitly, but the period in these cases is at most 6. These examples were added to in a paper by Yamamoto ([17]) and later, in [1], Bernstein obtained a large number of such polynomials f for which the continued fraction expansion of $\sqrt{f}$ could be made arbitrarily long and in [2] he gave explicit expressions for the fundamental unit in the quadratic field $\mathbb{Q}(\sqrt{f})$. Further examples were given in papers by Levesque and Rhin [5], Madden [6], Van Der Poorten [15] and Van Der Poorten and Williams [16].

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Some applications to determining the fundamental unit in real quadratic fields are also discussed.

Definition: A polynomial $f(t) \in Z[\,t\,]$ is called a *Fermat-Pell polynomial in one variable* if there exists polynomials $c(t), h(t) \in Z[\,t\,]$ such that

$$c(t)^2 - f(t)h(t)^2 = 1, \text{ for all t.} \tag{1.1}$$

A triple of polynomials $(c(t), h(t), f(t))$ satisfying equation 1.1 constitutes a *polynomial solution* to Pell's equation. A Fermat-Pell polynomial is said to have a *polynomial continued fraction expansion* if there exists an integer T and a positive integer n such that

$$\sqrt{f(t)} = [\,a_0(t); \overline{a_1(t), \cdots, a_n(t)}\,], \tag{1.2}$$

for all integral $t \geq T$, where the $a_i(t) \in \mathbb{Z}[\,t\,]$. For a fixed Fermat-Pell polynomial $f(t)$ a pair of polynomials $(c(t), h(t))$ is said to be the *fundamental polynomial solution* to (1.1) if $(c(t), h(t))$ constitutes the smallest solution in positive integers to equation 1.1 for all non-negative integral values of $t \geq T$.

In this paper it is shown how to construct several infinite families of Fermat-Pell polynomials in which the continued fraction expansion of the polynomials can have arbitrary long period length. Moreover, the polynomial continued fraction expansion of these polynomials can be written down explicitly.

2. Notation/ useful facts about convergents

As usual $[a_0; \overline{a_1, \cdots, a_n, 2a_0}]$ will denote the simple infinite periodic continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots\, a_{n-1} + \cfrac{\ddots}{a_n + \cfrac{1}{2a_0 + \cfrac{1}{a_1 + \ddots}}}}}.$$

$\dfrac{A_i}{B_i}$ will denote the i'th convergent:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots \cfrac{1}{a_i}}}.$$

Use will also be made of the following:

$$\begin{aligned} &A_i = a_i A_{i-1} + A_{i-2}, \\ &B_i = a_i B_{i-1} + B_{i-2}, \\ &A_i B_{i-1} - A_{i-1} B_i = (-1)^{i-1}, \text{ where} \\ &A_{-1} = B_{-2} = 1,\ A_{-2} = B_{-1} = 0, \end{aligned} \tag{2.1}$$

each of these being valid for $i = 0, 1, 2 \cdots$.

3. Some necessary Lemmas

Before coming to the main results of the paper several lemmas from the theory of the convergents of the continued fraction expansion of $\sqrt{f}$, where f is a non-square positive integer, are needed. Some are given without proof since the results are well known or are straightforward. In what follows c and h will denote the smallest pair of positive integers satisfying $c^2 - fh^2 = 1$. Recall that if the continued fraction expansion of $\sqrt{f}$ has period $2m$ then $(c, h) = (A_{2m-1}, B_{2m-1})$ and that if the continued fraction expansion of $\sqrt{f}$ has period $2m+1$ then $(c, h) = (A_{4m+1}, B_{4m+1})$.

Lemma 1. *Denote the i'th convergent to the continued fraction* $b_0 + \dfrac{1}{b_1+} \dfrac{1}{b_2 + \dots}$ *by* $b_0 + \dfrac{1}{b_1+} \dfrac{1}{b_2+} \cdots + \dfrac{1}{b_i} = \dfrac{P_i}{Q_i}$. *Then*

$$b_m + \frac{1}{b_{m-1}+} \frac{1}{b_{m-2}+} \cdots + \frac{1}{b_0} = \frac{P_m}{P_{m-1}}, \textit{ for all } m \geq 1. \tag{3.1}$$

Proof. This is well known. □

Label the usual three sequences of positive integers used to determine the continued fraction expansion $[a_0; a_1, \cdots]$ of $\sqrt{f}$ as follows:
$r_0 = 0, s_0 = 1, a_0 = \lfloor \sqrt{f} \rfloor$, $r_{k+1} = a_k s_k - r_k$, $s_{k+1} = (f - r_{k+1}^2)/s_k$ and $a_{k+1} = \lfloor (\sqrt{f} + r_{k+1})/s_{k+1} \rfloor$, for $k \geq 0$.

Lemma 2.

$$A_k A_{k-1} - f B_k B_{k-1} = (-1)^k r_{k+1}, \text{ for } k \geq -1. \tag{3.2}$$

$$A_k^2 - f B_k^2 = (-1)^{k+1} s_{k+1}, \text{ for } k \geq -1. \tag{3.3}$$

Proof. A simple double induction. □

Lemma 3. $r_k \leq a_0$, *for all* $k \geq 0$.

Proof. $r_{k+1} = a_k s_k - r_k = \lfloor (\sqrt{f} + r_k)/s_k \rfloor s_k - r_k < ((\sqrt{f} + r_k)/s_k) s_k - r_k$ $= \sqrt{f} < a_0 + 1$. □

Lemma 4. *If the continued fraction expansion of* $\sqrt{f}$ *has period* $2m$ *and* $a_m = a_0$ *or* $a_0 - 1$ *then* $s_m = 2$.

Proof. By Lemma 3 and the definition of a_0, $a_m = \lfloor (\sqrt{f} + r_m)/s_m \rfloor <$ $(\sqrt{f} + r_m)/s_m < (2a_0 + 1)/s_m$. If $a_m = a_0$ then $a_0 < (2a_0 + 1)/s_m$ and the fact that $r_m \neq 0$ clearly imply $s_m = 2$ (and $r_m = a_0$). The case for $a_m = a_0 - 1$ is similar (with $r_m = a_0 - 1$). □

Lemma 5. *If the continued fraction expansion of* $\sqrt{f}$ *has period of even length, say equal to* $2m$, *then*

$$c = A_m B_{m-1} + A_{m-1} B_{m-2}, \quad h = B_{m-1}(B_m + B_{m-2}).$$

Proof. By the symmetry of the continued fraction expansion of $\sqrt{f}$

$$[a_1, a_2, \cdots, a_{2m-2}, a_{2m-1}] = [a_1, a_2, \cdots, a_{m-1}, a_m, a_{m-1}, \cdots, a_2, a_1].$$

We know that $c = A_{2m-1}$, $h = B_{2m-1}$ and by the correspondence between matrices and convergents:

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{2m-1} & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} A_{2m-1} & A_{2m-2} \\ B_{2m-1} & B_{2m-2} \end{pmatrix},$$
$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_m & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} A_m & A_{m-1} \\ B_m & B_{m-1} \end{pmatrix},$$
$$\begin{pmatrix} a_{m-1} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} B_{m-1} & A_{m-1} - a_0 B_{m-1} \\ B_{m-2} & A_{m-2} - a_0 B_{m-2} \end{pmatrix},$$

$$\text{so that } \begin{pmatrix} A_{2m-1} & A_{2m-2} \\ B_{2m-1} & B_{2m-2} \end{pmatrix} = \begin{pmatrix} A_m & A_{m-1} \\ B_m & B_{m-1} \end{pmatrix} \begin{pmatrix} B_{m-1} & A_{m-1} - a_0 B_{m-1} \\ B_{m-2} & A_{m-2} - a_0 B_{m-2} \end{pmatrix}$$

and the result follows. □

Remark: One can show in essentially the same way that if the period of the continued fraction expansion of $\sqrt{f}$ is odd, say $2m+1$, then

$$\begin{aligned}c = A_{4m+1} &= (A_m^2 + A_{m-1}^2)(B_m^2 + B_{m-1}^2) + (A_m B_m + A_{m-1}B_{m-1})^2 \\ &= 2(A_m B_m + A_{m-1}B_{m-1})^2 + 1 \text{ and} \\ h = B_{4m+1} &= 2(A_m B_m + A_{m-1}B_{m-1})(B_m^2 + B_{m-1}^2).\end{aligned}$$

Lemma 6. *If the continued fraction expansion of $\sqrt{f}$ has period of even length which is congruent to $2 \mod 4$, say equal to $2m$, then*

$$hA_{m-1} - (c-1)B_{m-1} = 0.$$

Proof. By Lemma 5:

$$\begin{aligned}hA_{m-1} - (c-1)B_{m-1} &= B_{m-1}(B_m + B_{m-2})A_{m-1} \\ &\quad - (A_m B_{m-1} + A_{m-1}B_{m-2} - 1)B_{m-1} \\ = B_{m-1}(a_m B_{m-1} &+ B_{m-2} + B_{m-2})A_{m-1} \\ &\quad - ((a_m A_{m-1} + A_{m-2})B_{m-1} + A_{m-1}B_{m-2} - 1)B_{m-1} \\ = 2A_{m-1}B_{m-1}B_{m-2} &- (A_{m-2}B_{m-1}^2 + A_{m-1}B_{m-1}B_{m-2} - B_{m-1}) \\ = B_{m-1}(A_{m-1}B_{m-2} &- A_{m-2}B_{m-1} + 1) = B_{m-1}((-1)^{m-2} + 1) = 0.\end{aligned}$$

□

Lemma 7. *If $\sqrt{f} = [a_0; \overline{a_1, \cdots, a_n, 2a_0}]$ then $A_n = a_0 B_n + B_{n-1}$.*

Proof. Case(1) (the period is even, $= 2m$, say): By Lemma 5,

$$\begin{aligned}A_n = A_m B_{m-1} + A_{m-1}B_{m-2},\ B_n = B_{m-1}(B_m + B_{m-2}) \text{ and} \\ B_{n-1} = B_m(A_{m-1} - a_0 B_{m-1}) + B_{m-1}(A_{m-2} - a_0 B_{m-2})\end{aligned}$$

and the result follows easily.

Case(2) (The period is odd): Suppose the period is odd, $= 2m+1$, say and
$\sqrt{f} = [a_0; \overline{a_1, \cdots, a_m, a_m, \cdots, a_1, 2a_0}]$. Then

$$\begin{pmatrix} A_n & A_{n-1} \\ B_n & B_{n-1} \end{pmatrix} = \begin{pmatrix} A_m & A_{m-1} \\ B_m & B_{m-1} \end{pmatrix} \begin{pmatrix} B_m & A_m - a_0 B_m \\ B_{m-1} & A_{m-1} - a_0 B_{m-1} \end{pmatrix}$$

$A_n = A_m B_m + A_{m-1}B_{m-1}, \quad B_n = B_m^2 + B_{m-1}^2$,
$B_{n-1} = B_m(A_m - a_0 B_m) + B_{m-1}(A_{m-1} - a_0 B_{m-1})$ and again the result follows by simple arithmetic. □

Lemma 8. *If $\sqrt{f} = [a_0; \overline{a_1, \cdots, a_n, 2a_0}]$ then $B_n(f - a_0^2) = A_{n-1} + a_0 B_{n-1}$.*

Proof. Recall that if the period is even (n odd) then $A_n^2 - fB_n^2 = 1$, that if the period is odd (n even) then $A_n^2 - fB_n^2 = -1$ and that $A_nB_{n-1} - A_{n-1}B_n =$

$(-1)^{n-1}$. Then

$$\begin{aligned}
&B_n(f-a_0^2) = A_{n-1} + a_0 B_{n-1}\\
&\iff B_n f - a_0(A_n - B_{n-1}) = A_{n-1} + a_0 B_{n-1}, \quad \text{(by Lemma 7)}\\
&\iff B_n f = a_0 A_n + A_{n-1}\\
&\iff B_n^2 f = a_0 A_n B_n + A_{n-1} B_n\\
&\iff A_n^2 + (-1)^n = a_0 A_n B_n + A_n B_{n-1} + (-1)^n\\
&\iff A_n = a_0 B_n + B_{n-1}
\end{aligned}$$

and the result is true by Lemma 7. □

Lemma 9. *If the continued fraction expansion of $\sqrt{f}$ has period of odd length, $= 2m+1$, say, then $c = 2A_{2m}^2 + 1$ and $h = 2A_{2m}B_{2m}$.*

Proof. Elementary. □

Lemma 10. *Let $\sqrt{f} = [a_0; \overline{a_1, \cdots, a_{m-1}, a_m, a_{m-1}, \cdots, a_1, 2a_0}]$, where $a_m = a_0$ or $a_0 - 1$. Then*

$$\begin{aligned}
&(i) A_{m-1} = B_m + B_{m-2}\\
&(ii)\, c - 1 = A_{m-1}(B_m + B_{m-2}), \text{ if } m \text{ is odd,}\\
&(iii)\, c + 1 = A_{m-1}(B_m + B_{m-2}), \text{ if } m \text{ is even.}
\end{aligned}$$

Proof. (i)By the symmetry of the sequence $\{ r_i \}$ $(r_{m+i} = r_{m-i+1})$, it follows that $r_{m+1} = r_m$. By Lemma 2 and Lemma 4

$$\begin{aligned}
&A_m A_{m-1} - f B_m B_{m-1} = (-1)^m r_{m+1} = (-1)^m a_m\\
&A_{m-1}^2 - f B_{m-1}^2 = (-1)^m s_m = 2(-1)^m\\
&\Longrightarrow f = \frac{A_{m-1}^2 + (-1)^{m-1} 2}{B_{m-1}^2} = \frac{A_m A_{m-1} + (-1)^{m-1} a_m}{B_m B_{m-1}}\\
&\Longrightarrow A_{m-1}^2 B_m + (-1)^{m-1} 2 B_m = A_m A_{m-1} B_{m-1} + (-1)^{m-1} a_m B_{m-1}\\
&\Longrightarrow A_{m-1}^2 B_m + 2B_m = A_{m-1}(A_{m-1} B_m + (-1)^{m-1}) + a_m B_{m-1}\\
&\Longrightarrow 2B_m = A_{m-1} + a_m B_{m-1}
\end{aligned}$$

The result follows from the recurrence relation for the $B_i's$.
(ii)By Lemma 5:

$$\begin{aligned}
c - 1 &= A_m B_{m-1} + A_{m-1} B_{m-2} - 1\\
&= A_{m-1} B_m + (-1)^{m-1} + A_{m-1} B_{m-2} - 1\\
&= A_{m-1} B_m + A_{m-1} B_{m-2}.
\end{aligned}$$

(iii)Again by Lemma 5:

$$\begin{aligned}
c + 1 &= A_m B_{m-1} + A_{m-1} B_{m-2} + 1\\
&= A_{m-1} B_m + (-1)^{m-1} + A_{m-1} B_{m-2} + 1\\
&= A_{m-1} B_m + A_{m-1} B_{m-2}.
\end{aligned}$$

□

Lemma 11. *Let* $\sqrt{f} = [\,a_0; \overline{a_1, \cdots, a_n, 2a_0}\,]$*. If* X *and* Y *are positive integers satisfying*

$$X^2 - fY^2 = \pm 1$$

then $(X, Y) = (A_{k(n+1)-1}, B_{k(n+1)-1})$*, for some positive integer* k*.*

Proof. This is well known (See, for example, [3], page 387). □

The way this lemma will be used is as follows: Suppose it is known that $X^2 - fY^2 = \pm 1$ and that the continued fraction expansion of X/Y is $[\,a_0; a_1, \cdots, a_n]$. By the lemma, $X/Y = A_{k(n+1)-1}/B_{k(n+1)-1}$, for some positive integer k, and thus that the finite continued fraction expansion $[\,a_0; a_1, \cdots, a_n]$ either contains at least one complete period of the continued fraction expansion of $\sqrt{f}$ (the case $k > 1$) or else is just one term short of a full period (the case $k = 1$), in which case the missing term is of course $2a_0$ and $\sqrt{f} = [\,a_0; \overline{a_1, \cdots, a_n, 2a_0}\,]$. In particular, if $a_i \neq 2a_0$ for some i, $1 \leq i \leq n$, then the sequence $a_0, a_1, \cdots, a_n$ cannot contain a full period in the continued fraction expansion of $\sqrt{f}$ (since a full period of course ends in the term $2\,a_0$). This implies that $k = 1$ and that the latter case holds.

In the following theorems $\sqrt{f} = [\,a_0; \overline{a_1, \cdots, a_n, 2a_0}\,]$, unless otherwise stated. As above, c and h are the smallest pair of positive integers satisfying $c^2 - fh^2 = 1$. The variable t will be assumed to be a real variable. Initially, in the theorems that follow, the results shall be shown to be true for non-negative integral t and follow for real t by continuation.

Theorem 1. *Let* $f(t) = h^2t^2 + 2ct + f$*.*
(i)If n *is even then*

$$\sqrt{f(t)} = [\,ht + a_0; \overline{a_1, \cdots, a_n, 2a_0, a_1, \cdots, a_n, 2(ht + a_0)}], \quad \textit{for all } t \geq 0.$$

(ii) If n *is odd then*

$$\sqrt{f(t)} = [\,ht + a_0; \overline{a_1, \cdots, a_n, 2(ht + a_0)}\,], \quad \textit{for all } t \geq 0.$$

(iii)In either case $X = h^2t + c$*,* $Y = h$ *constitute the fundamental solution to* $X^2 - f(t)Y^2 = 1$*.*

Proof. It can be easily checked that $(h^2t + c)^2 - f(t)h^2 = 1$.
(i)For n even

$$\begin{aligned}
&[\,ht + a_0; a_1, \cdots, a_n, 2a_0, a_1, \cdots, a_n] = [\,ht + a_0; a_1, \cdots, a_n, a_0 + \frac{A_n}{B_n}] \\
&= ht + \frac{\left(a_0 + \frac{A_n}{B_n}\right) A_n + A_{n-1}}{\left(a_0 + \frac{A_n}{B_n}\right) B_n + B_{n-1}} = ht + \frac{(a_0B_n + A_n)\, A_n + A_{n-1}B_n}{(a_0B_n + A_n)\, B_n + B_{n-1}B_n} \\
&= ht + \frac{2A_n^2 - 1}{2A_nB_n}, \text{ by Lemma 7 and (2.1),}
\end{aligned}$$

$$= ht + \frac{c}{h} = \frac{h^2t+c}{h}, \text{ by Lemma 9.}$$

Since $2(ht+a_0)$ is not in the sequence $\{a_1, \cdots, a_n, 2a_0, a_1, \cdots, a_n\}$ it follows from Lemma11 that $\sqrt{f(t)}$ has the form claimed and that (iii) holds in the case n is even.
(ii) The case for n odd follows similarly since

$$[\,ht + a_0; a_1, \cdots, a_n] = ht + A_n/B_n = ht + c/h,$$

by a remark preceding Lemma 1.

□

Theorem 2. *Let $f(t) = (c-1)^2h^2t^2 + 2(c-1)^2t + f$.*
(i)If n is even

$$\sqrt{f(t)} = [\,(c-1)ht + a_0; \overline{a_1, \cdots, a_n, 2(c-1)ht + 2a_0}].$$

(ii) If n is odd and $\sqrt{f} = [\,a_0; \overline{a_1, \cdots, a_{m-1}, a_m, a_{m-1}, \cdots, a_1, 2a_0}\,]$, where $a_m = a_0$ or $a_0 - 1$ and m is odd then

$$\sqrt{f(t)} =$$
$$[(c-1)ht + a_0; \overline{a_1, \cdots, a_{m-1}, (c-1)ht + a_m, a_{m-1}, \cdots, a_1, 2(c-1)ht + 2a_0}].$$

(iii) In either case $X = (c-1)h^4t^2 + 2(c-1)h^2t + c$, $Y = h^3t + h$ constitute the fundamental solution to $X^2 - f(t)Y^2 = 1$.

Proof. In (iii) straightforward calculation shows that the given expressions for X and Y do constitute *a* solution to $X^2 - f(t)Y^2 = 1$. What needs to be shown for (iii) is that these choices of X and Y give the *fundamental* solution.
(i)Recall that for n even, $A_n^2 - fB_n^2 = -1$. Notice that

$$[\,(c-1)ht + a_0; a_1, \cdots, a_n] = (c-1)ht + \frac{A_n}{B_n} = \frac{(c-1)htB_n + A_n}{B_n} \text{ and}$$
$$((c-1)htB_n + A_n)^2 - f(t)B_n^2 = -1,$$

by the remark above, the formula for $f(t)$ and Lemma 9. Therefore, by Lemma 11, $\sqrt{f(t)}$ has the form claimed and, by Lemma 9, the smallest solution to $X^2 - f(t)Y^2 = 1$ is given by

$$X = 2((c-1)htB_n + A_n)^2 + 1 = (c-1)h^4t^2 + 2(c-1)h^2t + c,$$
$$Y = 2((c-1)htB_n + A_n)B_n = h^3t + h.$$

(ii)For n odd

$$[\,(c-1)ht + a_0; a_1, \cdots, a_{m-1}, (c-1)ht + a_m, a_{m-1}, \cdots, a_1]$$
$$= [\,(c-1)ht + a_0; a_1, \cdots, a_{m-1}, (c-1)ht + \frac{B_m}{B_{m-1}}], \text{ by Lemma 1}$$

$$
\begin{aligned}
&= (c-1)ht + \frac{\left((c-1)ht + \frac{B_m}{B_{m-1}}\right) A_{m-1} + A_{m-2}}{\left((c-1)ht + \frac{B_m}{B_{m-1}}\right) B_{m-1} + B_{m-2}} \\
&= (c-1)ht + \frac{(c-1)htB_{m-1}A_{m-1} + B_mB_{m-1} + A_{m-2}B_{m-1}}{(c-1)htB_{m-1}^2 + B_mB_{m-1} + B_{m-2}B_{m-1}} \\
&= (c-1)ht + \frac{(c-1)h^2t + c}{h^3t+h}, \text{ by Lemmas 5 and 10 and (2.1)} \\
&= \frac{(c-1)h^4t^2 + 2(c-1)h^2t + c}{h^3t+h}.
\end{aligned}
$$

The results follow by Lemma 11.

□

Theorem 3. *Let* $f(t) = (c+1)^2h^2t^2 + 2(c+1)^2t + f$.
If n *is odd and* $\sqrt{f} = [\,a_0; \overline{a_1, \cdots, a_{m-1}, a_m, a_{m-1}, \cdots, a_1, 2a_0}\,]$ *, where* $a_m = a_0$ *or* $a_0 - 1$ *and* m *is even then*

$$\sqrt{f(t)} =$$

$$[(c+1)ht + a_0; \overline{a_1, \cdots, a_{m-1}, (c+1)ht + a_m, a_{m-1}, \cdots, a_1, 2(c+1)ht + 2a_0}]$$

and $X = (c+1)h^4t^2 + 2(c+1)h^2t + c$, $Y = h^3t + h$ *constitute the fundamental solution to* $X^2 - f(t)Y^2 = 1$.

Proof. The proof here is virtually identical to the proof of part (ii) of the theorem above, the only difference being that part (iii) of Lemma 10 is used instead of part (ii). □

Theorem 4. *Let* $f(t) = (c+1)^2h^2t^2 + 2(c^2-1)t + f$.
If n *is even then*

$$\sqrt{f(t)} = [\,(c+1)ht + a_0; \overline{a_1, \cdots, a_n, 2(c+1)ht + 2a_0}]$$

and

$$X = \frac{(c+1)^2}{c-1}h^4t^2 + 2(c+1)h^2t + c, \;\; Y = \frac{(c+1)}{c-1}h^3t + h$$

constitute the fundamental solution to $X^2 - f(t)Y^2 = 1$.

Proof. By Lemma 9 $c = 2A_n^2 + 1$ and $h = 2A_nB_n$ and so $B_n^2 = h^2/(2c-2)$. Also, $A_n^2 - fB_n^2 = -1$.

$$[\,(c+1)ht + a_0; a_1, \cdots, a_n] = (c+1)ht + \frac{A_n}{B_n} = \frac{(c+1)htB_n + A_n}{B_n},$$

$$
\begin{aligned}
\text{and } ((c+1)htB_n + A_n)^2 - f(t)B_n^2 &= 2tB_n(c+1)(A_nh - (c-1)B_n) - 1 \\
&= -1,
\end{aligned}
$$

by Lemma 9. Therefore the continued fraction expansion of $\sqrt{f(t)}$ has the form claimed and, again by Lemma 9, the fundamental solution to $X^2 - f(t)Y^2 = 1$ is given by

$$X = 2((c+1)htB_n + A_n)^2 + 1$$

$$= 2(c+1)^2h^2t^2B_n^2 + 4(c+1)htB_nA_n + 2A_n^2 + 1,$$
$$= \frac{(c+1)^2}{c-1}h^4t^2 + 2(c+1)h^2t + c.$$
$$Y = 2((c+1)htB_n + A_n)B_n = 2(c+1)htB_n^2 + 2A_nB_n = \frac{(c+1)}{c-1}h^3t + h$$

□

Theorem 5. *Let*
$f(t) = (c-1)^2h^6t^4 + 4(c-1)^2h^4t^3 + 6(c-1)^2h^2t^2 + 2(c-1)(2c-1)t + f.$
(i)If n is odd and $\sqrt{f} = [a_0; \overline{a_1, \cdots, a_{m-1}, a_m, a_{m-1}, \cdots, a_1, 2a_0}]$ where $a_m = a_0$ or $a_0 - 1$ and m is odd then

$$\sqrt{f(t)} = [(c-1)(h^3t^2 + 2ht) + a_0; \overline{a_1, \cdots, a_{m-1}, (c-1)ht + a_m, a_{m-1}, \cdots, a_1, 2(c-1)(h^3t^2 + 2ht) + 2a_0}]$$

(ii)If n is even then

$$\sqrt{f(t)} = [(c-1)(h^3t^2 + 2ht) + a_0; \overline{a_1, \cdots, a_n, 2(c-1)ht + 2a_0, a_1, \cdots, a_n, 2(c-1)(h^3t^2 + 2ht) + 2a_0}].$$

(iii) In either case $X = (c-1)h^6t^3 + 3(c-1)h^4t^2 + 3(c-1)h^2t + c$, $Y = h + h^3t$ constitute the fundamental solution to $X^2 - f(t)Y^2 = 1$.

Proof. (i) As in the proof of Theorem 2

$$[(c-1)(h^3t^2 + 2ht) + a_0; a_1, \cdots, a_{m-1}, (c-1)ht + a_m, a_{m-1}, \cdots, a_1]$$
$$= (c-1)(h^3t^2 + 2ht) + \frac{(c-1)h^2t + c}{h^3t + h}$$
$$= \frac{(c-1)h^6t^3 + 3(c-1)h^4t^2 + 3(c-1)h^2t + c}{h^3t + h}$$

The results follow from Lemma 11.
(ii) Similarly,

$$[(c-1)(h^3t^2 + 2ht) + a_0; a_1, \cdots, a_n, 2(c-1)ht + 2a_0, a_1, \cdots, a_n]$$
$$= [(c-1)(h^3t^2 + 2ht) + a_0; a_1, \cdots, a_n, 2(c-1)ht + a_0 + \frac{A_n}{B_n}]$$
$$= (c-1)(h^3t^2 + 2ht) + \frac{\left(2(c-1)ht + a_0 + \frac{A_n}{B_n}\right)A_n + A_{n-1}}{\left(2(c-1)ht + a_0 + \frac{A_n}{B_n}\right)B_n + B_{n-1}}$$
$$= (c-1)(h^3t^2 + 2ht) + \frac{(2(c-1)htB_n + a_0B_n + A_n)A_n + A_{n-1}B_n}{(2(c-1)htB_n + a_0B_n + A_n)B_n + B_{n-1}B_n}$$
$$= (c-1)(h^3t^2 + 2ht) + \frac{(c-1)h^2t + c}{h^3t + h}, \quad \text{by Lemmas 7 and 9.}$$

The remainder of the proof parallels part (i). □

4. Fundamental Units in Real Quadratic Fields

In many cases it is easy to use the theorems in this paper to write down the fundamental unit in a wide class of real quadratic fields. On page 119 of [8] one has the following statement of the relationship of fundamental units in $\mathbb{Q}(\sqrt{D})$, D a square-free positive rational integer, and the convergents in the continued fraction expansion for $\sqrt{D}$:

Theorem 6. *Let D be a square-free, positive rational integer and let $K = \mathbb{Q}(\sqrt{D})$. Denote by ϵ_0 the fundamental unit of K which exceeds unity, by s the period of the continued fraction expansion for $\sqrt{D}$, and by P/Q the $(s-1)$-th convergent of it.*

If $D \not\equiv 1 \mod 4$ or $D \equiv 1 \mod 8$, then

$$\epsilon_0 = P + Q\sqrt{D}.$$

However, if $D \equiv 5 \mod 8$, then

$$\epsilon_0 = P + Q\sqrt{D}.$$

or

$$\epsilon_0^3 = P + Q\sqrt{D}.$$

Finally, the norm of ϵ_0 is positive if the period s is even and negative otherwise.

We note first of all that the congruence class of f, mod 4, forces c and h to lie in certain congruence classes. We have the following table:

f	c	h
1 (mod 4)	$\pm$ 1 (mod 8)	0 (mod 4)
2 (mod 4)	$\pm$ 1 (mod 16)	0 (mod 4)
	$\pm$ 3 (mod 8)	2 (mod 4)
3 (mod 4)	$\pm$ 1 (mod 8)	0 (mod 4)
	0 (mod 2)	1 (mod 2)

Table 1

In what follows let f be a non-square positive integer and (c, h) be the smallest pair of positive integers satisfying $c^2 - fh^2 = 1$. We denote the Fermat-Pell polynomials from Theorems 1 - 5 as follows:

$$\begin{aligned}
f_1(t) &= h^2t^2 + 2ct + f, \\
f_2(t) &= (c-1)^2h^2t^2 + 2(c-1)^2t + f, \\
f_3(t) &= (c+1)^2h^2t^2 + 2(c+1)^2t + f, \\
f_4(t) &= (c+1)^2h^2t^2 + 2(c^2-1)t + f, \\
f_5(t) &= (c-1)^2h^2t^2(h^5t^2 + 4h^2t + 6) + 2(c-1)(2c-1)t + f.
\end{aligned} \tag{4.1}$$

We consider first the case $f \equiv 2 \mod 4$. From the table above, c has to be odd, h has to be even and thus $f_2(t)$, $f_3(t)$, $f_4(t)$ and $f_5(t)$ are $\equiv 2 \mod 4$, for all integral $t \geq 0$ and $f_1(t)$ is $\equiv 2 \mod 4$ for all *even* $t \geq 0$. Thus we can apply Theorem 6.

As an illustration, by Theorems 1 and 6, if $f(t) = 4h^2t^2 + 4ct + f$ is squarefree, then the fundamental unit in $\mathbb{Q}(\sqrt{f(t)})$ is $2h^2t + c + h\sqrt{f(t)}$. With $f = 22$, $c = 197$ and $h = 42$, $22 + 788\,t + 7056\,t^2$ is squarefree for $150,601$ of the values of t lying between 0 and 200,000. In particlar, it is squarefree for $t = 199,998$ so that the fundamental unit in $\mathbb{Q}(\sqrt{282234512826670})$ is immediately known to be $705593141 + 42\sqrt{282234512826670}$.

Remark: The data in this case suggests the following question:

$$\text{Is } \lim_{N->\infty} \frac{\#\{t \in \mathbb{N}, t \leq N : 22 + 788\,t + 7056\,t^2 \text{ is squarefree}\}}{N} = \frac{3}{4}?$$

The author is presently unable to provide the answer.

A similar situation holds for either of the two possibilities listed in Table 1 for the case $f \equiv 3 \mod 4$. In the first case ($c \equiv \pm 1 \mod 8$ and $h \equiv 0 \mod 4$), $f_2(t)$, $f_3(t)$, $f_4(t)$ and $f_5(t)$ take positive integral values which are $\equiv 3 \mod 4$ for all integral $t \geq 0$, while $f_1(t)$ takes on positive integral values which are $\equiv 3 \mod 4$ for all integral *even* $t \geq 0$. In the second case ($c \equiv 0 \mod 2$ and $h \equiv 1 \mod 2$), all of the polynomials take positive integral values which are $\equiv 3 \mod 4$ for all *even* integral $t \geq 0$. Theorem 6 can again be used to write down the fundamental unit in infinitely many real quadratic fields. We again consider $h^2t^2 + 2ct + f$, with t even, and take $f = 43$, $c = 3482$ and $h = 531$. It is found that $43 + 13928\,t + 1127844\,t^2$ is squarefree for 147,511 of the integers t lying between 0 and 200,000. In particlar, it is squarefree for $t = 199,999$ so that the fundamental unit in $\mathbb{Q}(\sqrt{45113311649113959})$ is immediately known to be $112783839560 + 531\sqrt{45113311649113959}$.

Lastly, we consider the case $f \equiv 1 \mod 4$. For $c \equiv 1 \mod 8$, $f_2(t)$, $f_3(t)$, $f_4(t)$ and $f_5(t) \equiv f \mod 8$, for all integral $t \geq 0$, while $f_1(t) \equiv f \mod 8$ for all integral $t \equiv 0 \mod 4$. For $c \equiv -1 \mod 8$, $f_2(t)$, $f_3(t)$ and $f_4(t) \equiv f \mod 8$, for all integral $t \geq 0$, $f_1(t) \equiv f \mod 8$ for all positive $t \equiv 0 \mod 4$ and $f_5(t) \equiv f \mod 8$ for all *even* $t \geq 0$. For the case $c \equiv 0 \pmod 2$ and $h \equiv 1 \pmod 2$, $f_1(t)$ and $f_5(t) \equiv f \mod 8$ for all positive $t \equiv 0 \mod 4$, $f_2(t)$ $f_3(t)$ and $f_4(t) \equiv f \mod 8$ for all positive *even* t. In all of these cases, if $f \equiv 1 \mod 8$ and the resulting polynomial is squarefree, then the fundamental unit in the corresponding quadratic field can be written down. This time we consider the polynomial $f_2(t) = (c-1)^2h^2t^2 + 2(c-1)^2t + f$. By Theorems 2 and 6, if the continued fraction expansion of $\sqrt{f}$ has even period and $f_2(t)$ is squarefree, then the fundamental unit in $\mathbb{Q}(\sqrt{f(t)})$ is $(c-1)h^4t^2 + 2(c-1)h^2t + c + (h^3t + h)\sqrt{f_2(t)}$. As an illustration we take $f = 57$ so that $c = 151$ and $h = 20$. $f(t)$ is found to be squarefree for 121,529 of the integers t lying between 0 and 130,000. In particular, it is squarefree for $t = 130,000$ so so that the fundamental unit in $\mathbb{Q}(\sqrt{152100005850000057})$ is immediately known to be $405600015600000151 + 1040000020\sqrt{152100005850000057}$.

5. Concluding Remarks

In this paper only some limited classes of Fermat-Pell polynomials were considered (for example in all cases all but the end and possibly the middle terms in the continued fraction expansion were constant and the highest degree considered was 4). In a later paper I will consider multi-variable Fermat-Pell polynomials, polynomials whose continued fraction expansion has all terms non-constant and single-variable Fermat-Pell polynomials where the degree can be arbitrarily large.

Some problems still remain for the classes of polynomials examined here. Each of the triple of polynomials examined here constitute a polynomial solution to Pell's by virtue of the fact that $c^2 - fh^2 = 1$ and this alone. A natural question is to determine the continued fraction expansion of $f(t)$ in each of the cases examined when (c, h), (the smallest pair of positive integers satisfying the above integral Pell's equation) is replaced by the nth largest pair of such integers.

The continued fraction expansion of some of polynomials examined were given in some cases only in special circumstances. For example the continued fraction expansion of $f(t) = (c+1)^2h^2t^2 + 2(c+1)^2t + f$, examined in Theorem 3, was given only in the case where

$$\sqrt{f} = [\,a_0; \overline{a_1, \cdots, a_{m-1}, a_m, a_{m-1}, \cdots, a_1, 2a_0}] \text{ where } a_m = a_0 \text{ or } a_0 - 1$$

and m is even. There remains the problem of determining the continued fraction expansion of $f(t)$ when the length of the period of the continued fraction expansion of $\sqrt{f}$ is $\not\equiv 0 \mod 4$ or $a_m \neq a_0$ or $a_0 - 1$. Similar problems remain with the continued fraction expansion of the polynomials examined in some of the other theorems.

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Mathematics Department, University of Illinois, Champaign - Urbana, Illinois 61820

E-mail address: `jgmclaug@math.uiuc.edu`