McKibben Webster -- Chapter 1 Partial Solutions and Hints

Exercise 1.3.1:

$u''(t) + 2u'(t) - 3u(t) = 0$

$u(t) = Ce^{-3t}$

$u'(t) = -3Ce^{-3t}$

$u''(t) = 9Ce^{-3t}$

$(9Ce^{-3t}) + 2(-3Ce^{-3t}) - 3(Ce^{-3t}) = (9Ce^{-3t}) - 6Ce^{-3t} - 3Ce^{-3t} = 0$, as desired.

$Dom(u(t)) = (-\infty, \infty)$

Exercise 1.3.2:

$x^2 z''(x) + xz'(x) + z(x) = 0$

$z(x) = C_1 \cos(\ln x) + C_2 \sin(\ln x)$

$$z'(x) = \frac{-C_1 \sin(\ln x) + C_2 \cos(\ln x)}{x}$$

$$z''(x) = \frac{C_1 \sin(\ln x) - C_2 \sin(\ln x) - C_2 \cos(\ln x) - C_1 \cos(\ln x)}{x^2}$$

$$x^2 \left(\frac{C_1 \sin(\ln x) - C_2 \sin(\ln x) - C_2 \cos(\ln x) - C_1 \cos(\ln x)}{x^2}\right) +$$
$$x\left(\frac{-C_1 \sin(\ln x) + C_2 \cos(\ln x)}{x}\right) + C_1 \cos(\ln x) + C_2 \sin(\ln x) =$$

$$C_1 \sin(\ln x) - C_2 \sin(\ln x) - C_2 \cos(\ln x) - C_1 \cos(\ln x) - C_1 \sin(\ln x) + C_2 \cos(\ln x) +$$
$$C_1 \cos(\ln x) + C_2 \sin(\ln x) = 0$$

as desired.

$Dom(z(x)) = (0, \infty)$

Exercise 1.5.1:

$$y(t) = Ce^{kt}$$
$$y(3) = 400$$
$$y(10) = 2000$$
$$C = 400e^{-3k}$$
$$2000 = 400e^{-3k}e^{10k}$$
$$\frac{2000}{400} = e^{-3k+10k} = e^{7k}$$
$$5 = e^{7k}$$
$$\ln 5 = 7k$$
$$k = \frac{\ln 5}{7}$$

So, the general solution is $y(t) = Ce^{(\ln 5/7)t}$. Then to find C, proceed as follows:

$$400 = Ce^{(\ln 5/7)3}$$
$$400e^{-(\ln 5/7)3} = C$$

So, $y(t) = 400e^{-(\ln 5/7)3}e^{(\ln 5/7)t}$ is the solution of the IVP.

To answer the question, compute $y(0)$.

Matlab-Exercise 1.5.1:

i) $x(5)$=633.5278

ii)

$$405 = Ce^{3k}$$
$$2100 = Ce^{10k}$$
$$C = 405e^{-3k}$$
$$2100 = 405e^{10k}e^{-3k} = 405e^{10k-3k} = 405e^{7k}$$
$$\frac{2100}{405} = e^{7k}$$
$$k = \frac{\ln(5.185185185185185)}{7} = 0.2351150795$$
$$C = 405e^{-7(0.2351150795)} = 200.0448974504$$

iii) They look very similar. $x_2(t)$ starts to pull away at around t = 1.5.

STOP (page 10)

Here is the verification:

$$\frac{dT(x)}{dt} = k(T(x) - T_m)$$
$$dT(x) = k(T(x) - T_m)dt$$
$$\int_0^t \frac{dT(x)}{(T(x) - T_m)} = \int_0^t kdt$$
$$\ln(T(t) - T_m) - \ln(T_0 - T_m) = \ln(\frac{T(t) - T_m}{T_0 - T_m}) = kt$$
$$(\frac{T(t) - T_m}{T_0 - T_m}) = e^{kt}$$
$$T(t) = T_m + (T_0 - T_m)e^{kt}$$

Exercise 1.5.2:

$$T(t) = 5 + (T_0 - 5)e^{kt}$$
$$55 = 5 + (T_0 - 5)e^{k}$$
$$30 = 5 + (T_0 - 5)e^{5k}$$
$$50 = (T_0 - 5)e^{k}$$
$$25 = (T_0 - 5)e^{5k}$$
$$T_0 = 50e^{-k} + 5$$
$$25 = (50e^{-k} + 5 - 5)e^{5k} = 50e^{-k}e^{5k} = 50e^{5k-k} = 50e^{4k}$$
$$\frac{25}{50} = 0.5 = e^{4k}$$
$$k = \frac{\ln(0.5)}{4} = -0.1732867951$$
$$T_0 = 50e^{0.1732867951} + 5 = 64.4603557501$$

MatLab-Exercise 1.5.2:

i) T(8)=19.8651

ii)

$$55.55 = 5 + (T_0 - 5)e^{k}$$

$$30.3 = 5 + (T_0 - 5)e^{5k}$$

$$T_0 = 50.55e^{-k} + 5$$

$$30.3 = 5 + (50.55e^{-k} + 5 - 5)e^{5k} = 5 + 50.55e^{5k-k} = 5 + 50.55e^{4k}$$

$$25.3 = 50.55e^{4k}$$

$$k = \frac{\ln(\frac{25.3}{50.55})}{4} = -0.1730396375$$

$$T_0 = 50.55e^{0.1730396375} + 5 = 65.0995637773$$

iii) They start slightly apart and come together gradually

iv) They converge to the same point