MAT 162—Exam #3—11/22/11

Name: Solutions

Show all work using correct mathematical notation. Calculators are not permitted.

1. (12 points) Find the limit of each of the following sequences.

(a) $a_n = \dfrac{e^{3n}+4}{e^{3n+1}+5}$

$$\lim_{n\to\infty} a_n \underset{L'H}{=} \lim_{n\to\infty} \frac{3e^{3n}}{3e^{3n+1}} = \frac{1}{e}$$

(b) $a_n = \ln(5n^2+1) - \ln(n^2+3n+2)$ $= \ln\left(\dfrac{5n^2+1}{n^2+3n+2}\right)$

$$\lim_{n\to\infty} \frac{5n^2+1}{n^2+3n+2} = 5$$

So $\lim_{n\to\infty} a_n = \ln 5$

2. (13 points) In each case, find the sum of the series or show that the series diverges.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{5}{3^n}$ $= \dfrac{5}{1-1/3} = \dfrac{15}{2}$

geometric:

$c = 5$

$r = 1/3$

(b) $\displaystyle\sum_{n=4}^{\infty} \left(\cos(\tfrac{\pi}{n}) - \cos(\tfrac{\pi}{n+1})\right)$ $= \dfrac{\sqrt{2}}{2} - 1$

$$S_N = \left(\cos\frac{\pi}{4} - \cos\frac{\pi}{5}\right) + \left(\cos\frac{\pi}{5} - \cos\frac{\pi}{6}\right) + \left(\cos\frac{\pi}{6} - \cos\frac{\pi}{7}\right) + \cdots$$
$$+ \left(\cos\frac{\pi}{n-1} - \cos\frac{\pi}{n}\right) + \left(\cos\frac{\pi}{n} - \cos\frac{\pi}{n+1}\right)$$

$$= \cos\frac{\pi}{4} - \cos\frac{\pi}{n+1} \longrightarrow \cos\frac{\pi}{4} - \cos 0 \quad \text{as } n\to\infty$$

3. (25 points) Decide whether each series is convergent or divergent, and justify your answers using appropriate tests. You must give coherent arguments to receive credit.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{n^2+1}{n^3-n}$

We have $\dfrac{n^2+1}{n^3-n} \geq \dfrac{n^2}{n^3} = \dfrac{1}{n}$ so the series

diverges by direct comparison with a p-series $(p=1)$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{7^{2n}}{\sqrt{n!}}$

$a_n = \dfrac{7^{2n}}{\sqrt{n!}}$ $\quad a_{n+1} = \dfrac{7^{2n+2}}{\sqrt{(n+1)!}}$

$$\rho = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{7^{2n+2}}{\sqrt{(n+1)!}} \cdot \frac{\sqrt{n!}}{7^{2n}}$$

$$= \lim_{n \to \infty} \frac{49}{\sqrt{n+1}} = 0 < 1$$

so the series converges by the Ratio Test.

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{3/2}}$

$$\int_2^{\infty} \frac{1}{x(\ln x)^{3/2}}\,dx$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$= \lim_{b \to \infty} \int_2^b \frac{1}{x(\ln x)^{3/2}}\,dx$$

$$= \lim_{b \to \infty} \int_{\ln 2}^{\ln b} u^{-3/2}\,du$$

$$= \lim_{b \to \infty} -2u^{-1/2} \Big|_{\ln 2}^{\ln b}$$

$$= \lim_{b \to \infty} \left(\frac{-2}{\sqrt{\ln b}} + \frac{2}{\sqrt{\ln 2}} \right) = \frac{2}{\sqrt{\ln 2}}$$

So the series converges by the Integral Test.

4. (10 points) Let $a_n = \dfrac{n+7}{3n+5}$. Evaluate

(a) $\lim_{n\to\infty} a_n$

$= \frac{1}{3}$

(b) $\sum_{n=1}^{\infty} a_n$

$= \infty$ by Divergence Test since $\lim_{n\to\infty} a_n \neq 0$.

5. (15 points) Consider the series $S = \sum_{n=3}^{\infty} \dfrac{(-1)^{n-1}}{\ln(\ln n)}$.

(a) Show that the series converges conditionally. You must give a clear and complete argument, citing any appropriate tests.

Let $a_n = \dfrac{1}{\ln(\ln n)}$. Clearly $a_n > 0$ for $n \geq 3$, $a_{n+1} \leq a_n$, and $\lim_{n\to\infty} a_n = 0$. Hence the series converges by the AST.

However, $\lim_{n\to\infty} \dfrac{\ln(\ln n)}{n} \overset{L'H}{=} \lim_{n\to\infty} \dfrac{\frac{1}{\ln n}\cdot\frac{1}{n}}{1} = 0$,

so $\ln(\ln n) < n$ and hence $\dfrac{1}{\ln(\ln n)} > \dfrac{1}{n}$ for large n.

Therefore $\sum_{n=3}^{\infty} \dfrac{1}{\ln(\ln n)}$ diverges by limit comparison with a p-series.

(b) Let $S_N = \sum_{n=3}^{N} \dfrac{(-1)^{n-1}}{\ln(\ln n)}$. How large must N be to ensure that $|S - S_N| < \frac{1}{10}$?

We need $a_{N+1} = \dfrac{1}{\ln(\ln(N+1))} < \dfrac{1}{10}$

$\Leftrightarrow \ln(\ln(N+1)) > 10$

$\Leftrightarrow \ln(N+1) > e^{10}$

$\Leftrightarrow N > e^{e^{10}} - 1$

6. (10 points) Decide whether each statement is true or false. If a statement is false, give an example to show why.

(a) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

False : consider $a_n = \frac{1}{n}$

(b) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} |a_n|$ converges.

False : consider $a_n = \frac{(-1)^n}{n}$

7. (15 points) Find the radius and interval of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{(x+2)^n}{\sqrt{n}\,5^n}.$$

Justify your conclusions by citing appropriate tests.

Ratio Test : $\rho = \lim_{n\to\infty} \left| \frac{(x+2)^{n+1}}{\sqrt{n+1}\ 5^{n+1}} \cdot \frac{\sqrt{n}\ 5^n}{(x+2)^n} \right|$

$= \lim_{n\to\infty} \left| \frac{x+2}{5} \cdot \sqrt{\frac{n}{n+1}} \right| = \frac{|x+2|}{5}$

So the series converges absolutely when $|x+2| < 5$,

and we have $R = 5$.

Endpoints : $\underline{x = 3}$: $\sum_{n=1}^{\infty} \frac{5^n}{\sqrt{n}\ 5^n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ divergent p-series ($p = 1/2$)

$\underline{x = -7}$; $\sum_{n=1}^{\infty} \frac{(-5)^n}{\sqrt{n}\ 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by AST

Thus $I = [-7, 3)$.