MAT 162—Exam #3—11/18/15

Name: Solutions

Show all work using correct mathematical notation. Calculators are not permitted.

1. (15 points) Find the sum of each series, or demonstrate that it diverges.

(a) $\sum_{n=0}^{\infty} \frac{2^{3n+1}}{3^{2n+1}}$ $= \frac{2}{3} + \frac{16}{27} + \frac{2^7}{3^5} + \frac{2^{10}}{3^7} + \cdots$

geometric $= \frac{2/3}{1 - 8/9}$

$a = 2/3$

$r = 8/9$ $= 6$

(b) $\sum_{n=3}^{\infty} \left(\sqrt{n} - \sqrt{n+1}\right)$

$S_N = (\sqrt{3} - \sqrt{4}) + (\sqrt{4} - \sqrt{5}) + (\sqrt{5} - \sqrt{6}) + \ldots + (\sqrt{N-1} - \sqrt{N}) + (\sqrt{N} - \sqrt{N+1})$

$= \sqrt{3} - \sqrt{N+1}$ telescoping !

$\lim_{N \to \infty} S_N = -\infty$ so the series diverges

2. (10 points) Consider the sequence $a_n = \frac{3n^5 + 2}{4n^5 + 7}$. Evaluate

(a) $\lim_{n \to \infty} a_n$

$= \frac{3}{4}$ by L'Hopital's Rule or leading coeffs

(b) $\sum_{n=1}^{\infty} a_n$ $= \infty$ by the Divergence Test

since $\lim_{n \to \infty} a_n \neq 0$

3. (10 points) Find the limit of the sequence $a_n = n^{5/n}$. Show your work using correct limit notation.

$$\ln a_n = \frac{5}{n} \ln n$$

So
$$\lim_{n \to \infty} \ln a_n = \lim_{n \to \infty} \frac{5 \ln n}{n}$$

$$\underset{L'H}{=} \lim_{n \to \infty} \frac{5 \cdot \frac{1}{n}}{1} = 0$$

and thus
$$\lim_{n \to \infty} a_n = e^0 = 1$$

4. (15 points) Consider the series $S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$.

(a) Determine whether the series converges absolutely, converges conditionally, or diverges. Justify your answer using appropriate tests.

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^3} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3}$$ is a convergent

p-series $(p = 3 > 1)$

So the series S converges absolutely

(b) Write out the fourth partial sum, S_4.

$$S_4 = 1 - \frac{1}{8} + \frac{1}{27} - \frac{1}{64}$$

(c) How large must N be to ensure that the error in approximating S by the Nth partial sum S_N is at most 10^{-6}?

$$|S - S_N| \leq \frac{1}{(N+1)^3} \leq 10^{-6}$$

$$\Rightarrow (N+1)^3 \geq 10^6$$

$$\Rightarrow N+1 \geq 100 \Rightarrow N \geq 99$$

5. (25 points) Decide whether each series converges or diverges, and justify your conclusions using appropriate tests. You must give coherent arguments to receive credit.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

$$\int_2^{\infty} \frac{1}{x \ln x}\, dx = \lim_{b \to \infty} \int_2^{b} \frac{1}{x \ln x}\, dx$$

$u = \ln x$

$du = \frac{1}{x}\, dx$

$$= \lim_{b \to \infty} \int_{\ln 2}^{\ln b} \frac{1}{u}\, du$$

$$= \lim_{b \to \infty} \left(\ln(\ln b) - \ln(\ln 2) \right)$$

$$= \infty$$

$\therefore$ series diverges by Integral Test

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{3 + \sin n}{\sqrt{n^5 + 7n + 4}}$

We have $-1 \le \sin n \le 1$

$\Rightarrow 2 \le 3 + \sin n \le 4$

and $\sqrt{n^5 + 7n + 4} \geq \sqrt{n^5} = n^{5/2}$

So $\dfrac{3 + \sin n}{\sqrt{n^5 + 7n + 4}} \le \dfrac{4}{n^{5/2}}$ and the series converges by comparison with a p-series $(p = 5/2 > 1)$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(3n)!}{5^{2n} (n!)^3}$

Ratio Test: $$\rho = \lim_{n \to \infty} \frac{(3n+3)!}{5^{2n+2} \left[(n+1)!\right]^3} \cdot \frac{5^{2n} (n!)^3}{(3n)!}$$

$$= \lim_{n \to \infty} \frac{(3n+3)(3n+2)(3n+1)}{5^2 (n+1)^3}$$

$$= \frac{27}{25} > 1 \quad \text{by leading coeffs}$$

Hence the series diverges.

6. (10 points) Consider the series $S = \sum_{n=1}^{\infty} a_n$, whose Nth partial sum is $S_N = \sum_{n=1}^{N} a_n$.

(a) Suppose that $S_N = 5 - \frac{1}{3^N}$ for all N. Find the sum of the infinite series S.

$$S = \lim_{N \to \infty} \left(5 - \frac{1}{3^N} \right) = 5$$

(b) Find a_3, the third term in the series.

$$a_3 = S_3 - S_2 = \left(5 - \frac{1}{27} \right) - \left(5 - \frac{1}{9} \right) = \frac{2}{27}$$

7. (15 points) Find the radius and interval of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{2^n (x-5)^n}{\sqrt{n+1}}.$$

Justify your conclusions by citing appropriate tests.

Ratio Test:

$$\rho = \lim_{n \to \infty} \left| \frac{2^{n+1} (x-5)^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{2^n (x-5)^n} \right|$$

$$= \lim_{n \to \infty} \left| 2(x-5) \sqrt{\frac{n+1}{n+2}} \right|$$

$$= 2|x-5| < 1 \iff |x-5| < \frac{1}{2}$$

$$\therefore R = \frac{1}{2}$$

Endpoints:

$x = 11/2$: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ diverges by LCT with p-series ($p = 1/2 \le 1$)

$x = 9/2$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ converges by AST

$$\therefore I = \left[9/2, 11/2 \right)$$