MAT 162—Exam #3A—11/21/13

Name: Solutions

Show all work using correct mathematical notation. Calculators are not permitted.

1. (12 points) Find the limit of each of the following sequences. If the limit does not exist, explain why.

(a) $a_n = \dfrac{7n^2+1}{4n^3+5}$

$$\lim_{n\to\infty} a_n \underset{L'H}{=} \lim_{n\to\infty} \frac{14n}{12n^2} = \lim_{n\to\infty} \frac{7}{6n} = 0$$

(b) $b_n = \dfrac{\ln n}{\ln(3n^2+5)}$

$$\lim_{n\to\infty} b_n \underset{L'H}{=} \lim_{n\to\infty} \frac{1/n}{\frac{6n}{3n^2+5}} = \lim_{n\to\infty} \frac{3n^2+5}{6n^2} = \frac{1}{2}$$

2. (13 points) In each case, find the sum of the series. If the series diverges, explain why.

(a) $\displaystyle\sum_{n=4}^{\infty} \pi^{3-2n} = \pi^{-5} + \pi^{-7} + \pi^{-9} + \pi^{-11} + \cdots$

geometric

$$= \frac{\pi^{-5}}{1-\pi^{-2}}$$

$c = \pi^{-5}$

$r = \pi^{-2}$

(b) $\displaystyle\sum_{n=2}^{\infty} \cos(1/n)$

$$\lim_{n\to\infty} \cos(1/n) = \cos 0 = 1 \neq 0$$

so the series diverges by the Test for Divergence.

3. (25 points) Decide whether each series converges absolutely, converges conditionally, or diverges, and justify your conclusions using appropriate tests. You must give coherent arguments to receive credit.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{5/7}}$

Let $a_n = \frac{1}{n^{5/7}} > 0$. Then $a_{n+1} < a_n$ and $\lim_{n \to \infty} a_n = 0$, so the series converges by the AST.

$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^{5/7}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{5/7}}$ is a divergent p-series ($p = 5/7 \leq 1$), so the original series converges conditionally.

(b) $\sum_{n=5}^{\infty} \frac{3+\sin n}{\sqrt{n^2-1}}$

We have $-1 \leq \sin n \leq 1$ for all n, so $2 \leq 3 + \sin n \leq 4$.

Furthermore, $\sqrt{n^2-1} \leq \sqrt{n^2} = n$, so we obtain

$$\frac{3+\sin n}{\sqrt{n^2-1}} \geq \frac{2}{n}.$$

Hence the series diverges by direct comparison with the p-series $\sum_{n=5}^{\infty} \frac{1}{n}$ ($p=1$).

(c) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

Consider $\int_2^{\infty} \frac{1}{x(\ln x)^3}\, dx$

$= \lim_{b \to \infty} \int_2^b \frac{1}{x(\ln x)^3}\, dx$

$u = \ln x$

$du = \frac{1}{x}\, dx$

$= \lim_{b \to \infty} \int_{\ln 2}^{\ln b} u^{-3}\, du$

$= \lim_{b \to \infty} \left. \frac{u^{-2}}{-2} \right|_{\ln 2}^{\ln b}$

$= \lim_{b \to \infty} -\frac{1}{2}\left((\ln b)^{-2} - (\ln 2)^{-2} \right) = \frac{1}{2(\ln 2)^2}$

Hence the series converges (absolutely) by the Integral Test.

4. (15 points) Consider the series $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$.

(a) Show (using an appropriate test) that the series converges absolutely.

The Ratio Test gives

$$\rho = \lim_{n\to\infty} \left| \frac{(-1)^{n+2}}{(n+1)!} \cdot \frac{n!}{(-1)^{n+1}} \right|$$

$$= \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1,$$

so the series converges absolutely.

(b) Write out the fourth partial sum, S_4.

$$S_4 = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24}$$

(c) If we use the approximation $S \approx S_4$, what is the maximum error in our estimate?

$$|S - S_4| < a_5 = \frac{1}{5!} = \frac{1}{120}$$

5. (10 points) Consider the following statement:

"If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges."

Either explain why the statement is true, or give a specific example to show that it is false.

The statement is false. Consider $a_n = \frac{1}{n}$.

Then $\lim_{n\to\infty} \frac{1}{n} = 0$ but $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent p-series.

6. (7 points) Find the sum of the series $\sum_{n=3}^{\infty}\left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$.

$$S_N = \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}}\right) + \left(\frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}}\right) + \left(\frac{1}{\sqrt{5}} - \frac{1}{\sqrt{6}}\right) + \left(\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{7}}\right) + \cdots$$

$$\cdots + \left(\frac{1}{\sqrt{N-2}} - \frac{1}{\sqrt{N-1}}\right) + \left(\frac{1}{\sqrt{N-1}} - \frac{1}{\sqrt{N}}\right) + \left(\frac{1}{\sqrt{N}} - \frac{1}{\sqrt{N+1}}\right)$$

$$= \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{N+1}}$$

Hence $\sum_{n=3}^{\infty}\left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) = \lim_{N\to\infty}\left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{N+1}}\right) = \frac{1}{\sqrt{3}}$

7. (18 points) Find the radius and interval of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{(x+8)^n}{n^2 5^{2n+1}}.$$

Justify your conclusions by citing appropriate tests.

Ratio Test :

$$\rho = \lim_{n\to\infty} \left| \frac{(x+8)^{n+1}}{(n+1)^2 5^{2n+3}} \cdot \frac{n^2 5^{2n+1}}{(x+8)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x+8}{5^2} \cdot \left(\frac{n}{n+1}\right)^2 \right| = \frac{|x+8|}{25}$$

So the series converges absolutely if $|x+8| < 25$ (that is, if $-33 < x < 17$) and diverges if $|x+8| > 25$.

It follows that $R = 25$.

Endpoints :

$x = 17$: $\sum_{n=1}^{\infty} \frac{25^n}{n^2 5^{2n+1}} = \sum_{n=1}^{\infty} \frac{1}{5n^2}$ converges $(p = 2 > 1)$

$x = -33$: $\sum_{n=1}^{\infty} \frac{(-25)^n}{n^2 5^{2n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{5n^2}$ converges by AST (or by absolute convergence)

Thus $I = [-33, 17]$.