MAT 261—Exam #2A—3/13/14

Name: Solutions

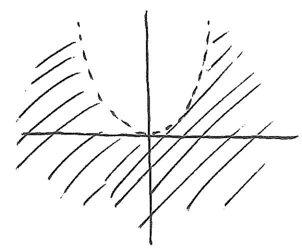
Calculators are not permitted. Show all of your work using correct mathematical notation.

1. (25 points) Consider the function $f(x, y) = \ln(x^2 - y)$.

(a) Sketch the domain of f.

$$x^2 - y > 0$$

$$\Leftrightarrow \quad y < x^2$$

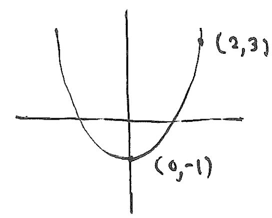


(b) Find the equation of the level curve passing through the point $(2, 3)$, and sketch its graph.

$$f(2,3) = \ln 1 = 0$$

$$\ln(x^2 - y) = 0$$

$$\Leftrightarrow \quad x^2 - y = 1 \quad \Leftrightarrow \quad y = x^2 - 1$$



(c) Find the average rate of change of f from $(2, 1)$ to $(2, 3)$.

$$\frac{f(2,3) - f(2,1)}{3 - 1} = \frac{-\ln 3}{2}$$

(d) Find the gradient of f at the point $(2, 1)$.

$$\vec{\nabla} f = \frac{2x}{x^2 - y}\,\hat{\imath} - \frac{1}{x^2 - y}\,\hat{\jmath}$$

$$\vec{\nabla} f_{(2,1)} = \frac{4}{3}\,\hat{\imath} - \frac{1}{3}\,\hat{\jmath}$$

(e) Find the instantaneous rate of change of f at the point $(2, 1)$ in the direction $\mathbf{j}$.

$$\vec{\nabla} f_{(2,1)} \cdot \hat{\jmath} = \left.\frac{\partial f}{\partial y}\right|_{(2,1)} = -\frac{1}{3}$$

2. (15 points) Let $f(x,y,z) = z^3\cos(xy^2) + e^{xyz}\tan z$. Calculate f_x, f_y, and f_z.

$$f_x = -y^2 z^3 \sin(xy^2) + yz\, e^{xyz} \tan z$$

$$f_y = -2xyz^3 \sin(xy^2) + xz\, e^{xyz} \tan z$$

$$f_z = 3z^2 \cos(xy^2) + e^{xyz} \sec^2 z$$

$$+ xy\, e^{xyz} \tan z$$

3. (10 points) Find the linearization of the function $f(x,y) = \sqrt{x^2+y^4}$ at the point $(3,2)$.

$f(3,2) = 5$

$$f_x = \frac{1}{2}(x^2+y^4)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2+y^4}}$$

$$\Rightarrow \quad f_x(3,2) = \frac{3}{5}$$

$$f_y = \frac{1}{2}(x^2+y^4)^{-1/2} \cdot 4y^3 = \frac{2y^3}{\sqrt{x^2+y^4}}$$

$$\Rightarrow \quad f_y(3,2) = \frac{16}{5}$$

Thus $L(x,y) = 5 + \frac{3}{5}(x-3) + \frac{16}{5}(y-2)$

4. (30 points) Consider the function $f(x, y) = x^2 + y^2 - xy + x$.

(a) Find the maximum value of the directional derivative of f at the point $(3, 5)$.

$$\vec{\nabla} f = (2x - y + 1)\,\hat{\imath} + (2y - x)\,\hat{\jmath}$$

$$\Rightarrow \vec{\nabla} f_{(3,5)} = 2\hat{\imath} + 7\hat{\jmath}$$

$$\therefore \|\vec{\nabla} f_{(3,5)}\| = \sqrt{53} \text{ is the maximum value}$$

(b) Find the directional derivative of f at the point $(3, 5)$ in the direction of $\mathbf{v} = 3\mathbf{i} - \mathbf{j}$.

$$\vec{u} = \vec{e}_{\vec{v}} = \frac{3\hat{\imath} - \hat{\jmath}}{\sqrt{10}}$$

$$D_{\vec{u}} f(3,5) = \vec{\nabla} f_{(3,5)} \cdot \vec{u}$$

$$= -\frac{1}{\sqrt{10}}$$

(c) Find the coordinates of all local maxima, local minima, and saddle points of f.

$$f_x = f_y = 0 \Rightarrow \begin{cases} 2x - y + 1 = 0 \\ 2y - x = 0 \end{cases} \Rightarrow x = 2y$$

$$\Rightarrow 3y + 1 = 0$$

$$\Rightarrow y = -1/3, \quad x = -2/3$$

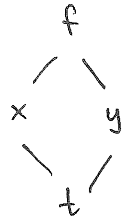
$$f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = -1$$

$$\Rightarrow D = 2 \cdot 2 - (-1)^2 = 3 > 0$$

Since $f_{xx} > 0$ we conclude that f has a local minimum at $(-2/3, -1/3)$.

(d) If $x = \sin 2t$ and $y = 2e^{3t}$, calculate $\dfrac{df}{dt}$ when $t = 0$.

$$t = 0 \Rightarrow (x, y) = (0, 2)$$



$$\frac{df}{dt} = (2x - y + 1) \cdot 2\cos 2t + (2y - x) \cdot 6e^{3t}$$

$$\Rightarrow \left.\frac{df}{dt}\right|_{t=0} = -1 \cdot 2 + 4 \cdot 6 = 22$$

5. (10 points) Let $f(x,y) = \dfrac{x^2y}{x^4+y^2}$. Show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Hint: Consider a parabolic path of approach.

Along the x-axis, we have $y=0$ and

$$f(x,0) = \frac{0}{x^4} = 0 \quad \text{for all } x \neq 0.$$

Along the parabola $y = x^2$ we have

$$f(x,x^2) = \frac{x^4}{x^4+x^4} = \frac{x^4}{2x^4} = \frac{1}{2} \quad \text{for all } x \neq 0.$$

Hence the Two Path Test implies that

$\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

6. (10 points) Use Lagrange multipliers to find the point on the line $4x - 6y = 25$ where the function $f(x,y) = x^2 + 2y^2$ has its minimum value.

$$g(x,y) = 4x - 6y = 25 \quad \text{constraint}$$

$$\vec{\nabla} f = 2x\,\hat{\imath} + 4y\,\hat{\jmath} \qquad \vec{\nabla} g = 4\hat{\imath} - 6\hat{\jmath}$$

$$\vec{\nabla} f = \lambda \vec{\nabla} g \Rightarrow \begin{cases} 2x = 4\lambda \\ 4y = -6\lambda \end{cases} \Rightarrow \lambda = \tfrac{1}{2}x$$

$$\Rightarrow 4y = -3x$$

$$\Rightarrow y = -\tfrac{3}{4}x$$

The constraint now gives

$$4x - 6\left(-\tfrac{3}{4}x\right) = 25$$

$$\Rightarrow \frac{17}{2}x = 25 \Rightarrow x = 50/17,$$

$$y = -75/34$$

Hence f attains its minimum value on the line at $(50/17, -75/34)$. Note that the line contains points with arbitrarily large x & y values, so there is no max.