ON SIMULTANEOUS DIAGONAL INEQUALITIES, III

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1. Introduction

The study of diophantine inequalities began with the work of Davenport and Heilbronn [8], who showed that an indefinite additive form with real coefficients takes arbitrarily small values infinitely often at integral points, provided that the number of variables is sufficiently large in terms of the degree. Their version of the Hardy-Littlewood method has been adapted to treat more general situations in the intervening years. For example, Schmidt [16] has obtained a result for arbitrary (not necessarily diagonal) forms of odd degree, and the problem for systems of forms of like degree has been examined by a number of workers (see for example [5], [6], [7], and [15]). More recently, the author [12] began investigating systems of inequalities of differing degree, starting with the case of one cubic and one quadratic form.

Throughout these efforts, the inability to adequately control the rational approximations to various coefficient ratios in the forms under consideration has resulted in somewhat weaker theorems than may have been expected. In the case of a single inequality, for example, it was not possible to obtain a lower bound for the number of solutions in a given box, since the parameter representing the box size was restricted in terms of a possibly sparse sequence of denominators occurring in a continued fraction expansion. In certain other situations (see [5], [7], [9], and [12]), a difficulty of similar spirit forced a restriction to forms with algebraic (or at least badly approximable) coefficient ratios. The past few years, however, have seen a remarkable breakthrough in this area, beginning with work of Bentkus and Götze [4] on values of positive-definite quadratic forms. Drawing inspiration from their methods, Freeman [10] was able to obtain the expected asymptotic lower bound for the number of solutions in the Davenport-Heilbronn problem. The author [13] then adapted Freeman's ideas to remove the restriction to algebraic coefficients for the system considered in [12]. It is now possible, using an even more recent result of Freeman [11], to extend the work of [12] and [13] to more general systems of diagonal inequalities.

Suppose that $k_1 > \cdots > k_t \geq 1$ are integers, let λ_{ij} be non-zero real numbers, and fix $\tau > 0$. We consider the system of inequalities

$$|\lambda_{i1}x_1^{k_i} + \cdots + \lambda_{is}x_s^{k_i}| < \tau \quad (1 \leq i \leq t). \tag{1.1}$$

Under certain conditions, we are able to demonstrate that the system (1.1) has infinitely many solutions in integers $x_1, \ldots, x_s$ and in fact that the number of such solutions lying in the box $[-P, P]^s$ is of order P^{s-K}, where $K = k_1 + \cdots + k_t$. Perhaps the condition of greatest interest in the statement of such a result is how large s is required to be in terms of $\mathbf{k} = (k_1, \ldots, k_t)$. In order to investigate such bounds for s, we need to

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impose some further conditions. To ensure indefiniteness, we require that the system of equations

$$\lambda_{i1}x_1^{k_i} + \cdots + \lambda_{is}x_s^{k_i} = 0 \qquad (1 \le i \le t) \tag{1.2}$$

possesses a non-trivial real solution. Moreover, if one or more of the forms appearing in (1.1) is a multiple of an integral form (*i.e.*, if all its coefficients are in rational ratio), then for τ sufficiently small, the corresponding inequalities are equivalent to homogeneous equations with integer coefficients, and this gives rise to a p-adic solubility requirement. In what follows, we refer to this set of equations as the *integral sub-system*. Furthermore, in order to apply the circle method successfully, one needs to ensure that the given local solutions are in fact non-singular. Therefore, we say that the system (1.1) satisfies the *local solubility condition* if each λ_{ij} is non-zero, the system (1.2) has a non-singular real solution, and the integral sub-system of (1.2) has a non-singular p-adic solution for every prime p. We then define $\widehat{G}^*(\mathbf{k})$ to be the least integer s_0 such that, whenever $s \ge s_0$ and the system (1.1) satisfies the local solubility condition, the system (1.1) has a non-trivial integral solution.

As is familiar in applications of the circle method, bounds for $\widehat{G}^*(\mathbf{k})$ can be expected to depend primarily on the quality of available estimates for certain exponential sums and their various moments. When P and R are positive numbers, let

$$\mathcal{A}(P,R) = \{n \in [1,P] \cap \mathbb{Z} : p|n,\ p \text{ prime} \Rightarrow p \le R\}$$

denote the set of R-smooth numbers not exceeding P. We shall be concerned with the exponential sums

$$F(\boldsymbol{\alpha}) = F(\boldsymbol{\alpha};P) = \sum_{1\le x\le P} e(\alpha_1 x^{k_1} + \cdots + \alpha_t x^{k_t}) \tag{1.3}$$

and

$$f(\boldsymbol{\alpha}) = f(\boldsymbol{\alpha};P,R) = \sum_{x\in\mathcal{A}(P,R)} e(\alpha_1 x^{k_1} + \cdots + \alpha_t x^{k_t}), \tag{1.4}$$

where we have written, as usual, $e(y) = e^{2\pi i y}$, and where R is taken to be a sufficiently small power of P. Using the technology developed by Wooley [20], one can obtain mean value estimates of the shape

$$\int_{\mathbb{T}^t} |f(\boldsymbol{\alpha})|^{2u}\, d\boldsymbol{\alpha} \ll P^{2u-K+\Delta_u}, \tag{1.5}$$

where $\Delta_u = \Delta_{u,\mathbf{k}}$ is small when u is sufficiently large in terms of $\mathbf{k}$, and where $\mathbb{T}^t$ denotes the t-dimensional unit cube. An elementary argument shows that one always has $\Delta_u \ge 0$ in (1.5). In our applications, we will often need to find u for which $\Delta_u = 0$.

Write $Q_i = 2k_1^2 P^{k_i-1}$, and let $\mathfrak{m}$ denote the set of $\boldsymbol{\alpha} \in \mathbb{T}^t$ such that whenever there are integers $q \ge 1$ and $a_1,\ldots,a_t$ with $(q,a_1,\ldots,a_t) = 1$ and $|q\alpha_i - a_i| < Q_i^{-1}$ for $i = 1,\ldots,t$, one has that $q > P$. We consider estimates for $F(\boldsymbol{\alpha})$ of the shape

$$\sup_{\boldsymbol{\alpha}\in\mathfrak{m}} |F(\boldsymbol{\alpha})| \ll P^{1-\sigma(\mathbf{k})+\varepsilon}, \tag{1.6}$$

where ε is an arbitrarily small positive number. It follows from work of Baker [1] (see also [2], [3]) that one can take $\sigma(\mathbf{k}) = 2^{1-k_1}$, while Vinogradov's methods (see for example [2], Theorem 4.4) show that one can take $\sigma(\mathbf{k}) \sim (8k_1^2 \log k_1)^{-1}$ for large k_1.

The purpose of this paper is to demonstrate that bounds for $\widehat{G}^*(\mathbf{k})$ do indeed follow from estimates of the type (1.5) and (1.6). As the quality of such estimates is likely to improve over time, we state our bounds in a form convenient for incorporating future developments into the problem at hand.

Theorem 1.1. *Suppose that $k_1 > \cdots > k_t \geq 1$. Let u and v be positive integers, and write $\gamma = 0$ if v is even and $\gamma = 1$ if v is odd. Further suppose that*

$$v \geq \max(2, t), \quad 2u + v > k_1(t+1)(1 + \gamma/v), \quad \text{and} \quad v\sigma(\mathbf{k}) > \Delta_u,$$

where Δ_u and $\sigma(\mathbf{k})$ satisfy (1.5) and (1.6). Then one has $\widehat{G}^*(\mathbf{k}) \leq 2u + v$.

In fact, one can give a similar statement that incorporates a Weyl-type bound for $f(\boldsymbol{\alpha})$ rather than $F(\boldsymbol{\alpha})$. We forego the statement of a general theorem along these lines but will revisit this issue in connection with Corollary 1.3 below.

By taking $\sigma(\mathbf{k}) = 2^{1-k_1}$ in (1.6) and using the values of Δ_u calculated by the author [14], one can obtain some explicit bounds for the case $t = 2$. We record the results for several interesting cases below.

Corollary 1.2. *One has the bounds*

$$\begin{gathered}\widehat{G}^*(3,2) \leq 13, \quad \widehat{G}^*(4,2) \leq 20, \quad \widehat{G}^*(4,3) \leq 24, \quad \widehat{G}^*(5,2) \leq 31,\\ \widehat{G}^*(5,3) \leq 32, \quad \widehat{G}^*(5,4) \leq 36, \quad \widehat{G}^*(6,3) \leq 49, \quad \widehat{G}^*(6,4) \leq 47,\\ \widehat{G}^*(6,5) \leq 50, \quad \widehat{G}^*(7,4) \leq 65, \quad \widehat{G}^*(7,5) \leq 64, \quad \widehat{G}^*(7,6) \leq 66.\end{gathered}$$

Note that these bounds are the same as those recorded in [14] for pairs of equations. In fact, the results given here for inequalities imply the results on equations, since we have not excluded the case where all the forms in (1.1) are multiples of integral forms.

Finally, we can apply the results of Wooley [20] to obtain more general bounds for large k_1. Here it transpires that, when t is small relative to k_1, one can obtain significantly better estimates of the form (1.6), but with $F(\boldsymbol{\alpha})$ replaced by $f(\boldsymbol{\alpha})$, by making use of the mean value estimates (1.5) within a large sieve argument. At the end of §3, we sketch a modification of the proof of Theorem 1.1 that yields the following bounds.

Corollary 1.3. *Suppose that $k_1 > \cdots > k_t \geq 1$, and define*

$$H(\mathbf{k}) = k_1 \min\left\{t(\log k_1 + 3\log t),\, 3t^2 + 6t\log\log k_1 + \log(k_1 \cdots k_t)\right\}.$$

Then one has $\widehat{G}^*(\mathbf{k}) \leq H(\mathbf{k})(1 + o(1))$. Moreover, if $t \geq \sqrt{k_1}$, then one has

$$\widehat{G}^*(\mathbf{k}) \leq tk_1\big(3\log k_1 - \log t + 4\log\log k_1 + O(1)\big).$$

For example, if t is bounded by a constant, then we can take $s \sim k_1 \log(k_1 \cdots k_t)$ as $k_1 \to \infty$, whereas we require $s \sim 2k_1^2 \log k_1$ if $t \sim k_1$. It may be possible to apply the methods of [20] to obtain even more precise estimates, but we do not pursue this here.

Note that in (1.1) we have restricted to the case in which all the coefficients λ_{ij} are non-zero. In practice, this condition can be relaxed somewhat (see for example [12], [13]), and for equations it can sometimes be removed completely (see [14], §8). In general, however, the analytic argument permits one to handle only a limited number of zero coefficients directly, and one typically has to obtain small solutions to various auxiliary systems in order to reduce to such a situation. Since the latter presents significant complications in its own right, we prefer not to deal with such issues here. We do point out that almost all systems of the shape (1.1) have no vanishing coefficients. Finally, we note that the role played here by zero coefficients is similar in some respects to that played by vanishing linear combinations in systems of forms of like degree.

While combining estimates of the shape (1.5) and (1.6) suffices to handle the minor arcs for the problem on equations, some additional maneuvering is required to deal with inequalities. In the latter case, one expects that the product of the exponential sums $F_j(\boldsymbol{\alpha}) = F(\boldsymbol{\lambda}_j\boldsymbol{\alpha})$ has only one substantial peak, at the origin, rather than about every rational point with sufficiently small common denominator; here we have written $\boldsymbol{\lambda}_j\boldsymbol{\alpha}$ for the vector $(\lambda_{1j}\alpha_1, \dots, \lambda_{tj}\alpha_t)$. Hence the minor arc region on which one must obtain non-trivial bounds for these sums is somewhat larger, and one must try to exploit the irrationality of various coefficient ratios to demonstrate the expected cancellation. This is precisely the point of difficulty alluded to above, which had caused problems for several generations of workers. The idea of Bentkus and Götze [4] is to aim for very weak versions of (1.6), valid on a carefully defined set of minor arcs, and then choose u so that $\Delta_u = 0$ in (1.5). If none of the $F_j(\boldsymbol{\alpha})$ were $o(P)$, then Baker's work would yield exceptionally good rational approximations to the $\lambda_{ij}\alpha_i$, and, if $\boldsymbol{\alpha}$ is not too close to the origin, one hopes that this would contradict the existence of an irrational ratio $\lambda_{ij}/\lambda_{ik}$. Baker's method actually fails by a factor of P^ε to produce the required bound on the denominators occurring in these approximations, but it turns out that this factor can be removed. This was first accomplished by Freeman, through a difficult argument, in an early version of his paper [11] on inhomogeneous inequalities, but the proof has been greatly simplified by a recent observation of Wooley.

Our strategy is to first use the estimates (1.5) and (1.6) to obtain new estimates of the shape (1.5), with $\Delta_u = 0$, by means of a Hardy-Littlewood dissection. The required analysis here largely resembles that of [14], §7, and hence the number of variables required to obtain this full savings will usually be the same as the corresponding bound in Theorem 1.1 of [14]. However, there is typically some additional room to spare in the minor arc contribution to these mean values, and thus we can in fact apply the dissection with a fractional number of variables, slightly less than the number one is expecting to use for the problem. One can then incorporate the weak Weyl estimate described in the previous paragraph to the remaining fraction of a sum and hence use exactly the same number of variables as for equations.

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2. Preliminary Estimates

We start by describing the Hardy-Littlewood dissection that will be employed in the deduction of our mean value estimates. As in §1, we write

$$Q_i = 2k_1^2 P^{k_i - 1} \quad (i = 1, \dots, t).$$

Then when $k_t > 1$, we define the major arcs $\mathfrak{M}$ to be the union of the boxes

$$\mathfrak{M}(q,\mathbf{a}) = \{\boldsymbol{\alpha} \in \mathbb{T}^t : |q\alpha_i - a_i| < Q_i^{-1} \quad (i = 1, \dots, t)\} \tag{2.1}$$

with $0 \le a_1, \dots, a_t \le q \le P$ and $(q, a_1, \dots, a_t) = 1$, and we write $\mathfrak{m} = \mathbb{T}^t \setminus \mathfrak{M}$ for the minor arcs. When $k_t = 1$, we use the same definition, except that $\mathfrak{M}$ is further restricted to those $\mathfrak{M}(q,\mathbf{a})$ for which $(q, a_1, \dots, a_{t-1}) \le P^{\varepsilon}$. Note that the $\mathfrak{M}(q,\mathbf{a})$ may not be disjoint when $k_t = 1$. We may clearly suppose throughout that $k_1 \ge 2$, as Theorem 1.1 is otherwise trivial. Let us introduce the notation

$$S(q,\mathbf{a}) = \sum_{x=1}^{q} e((a_1 x^{k_1} + \dots + a_t x^{k_t})/q),$$

$$v(\boldsymbol{\beta}) = \int_0^P e(\beta_1 \gamma^{k_1} + \dots + \beta_t \gamma^{k_t})\, d\gamma,$$

and

$$w(\boldsymbol{\beta}) = \int_R^P \rho\left(\frac{\log \gamma}{\log R}\right) e(\beta_1 \gamma^{k_1} + \dots + \beta_t \gamma^{k_t})\, d\gamma,$$

where ρ denotes Dickman's function (see for example Vaughan [18], §12.1). We first recall some standard estimates for these functions. It follows easily from Theorem 7.1 of Vaughan [18] that

$$S(q,\mathbf{a}) \ll (q, a_1, \dots, a_t)^{1/k_1} q^{1-1/k_1+\varepsilon}. \tag{2.2}$$

Moreover, by applying the argument of Vaughan [18], Theorem 7.3, as in the proof of Wooley [19], Lemma 8.6, one finds that

$$v(\boldsymbol{\beta}) \ll P(1 + P^{k_1}|\beta_1| + \dots + P^{k_t}|\beta_t|)^{-1/k_1} \tag{2.3}$$

and

$$w(\boldsymbol{\beta}) \ll P(1 + P^{k_1}|\beta_1| + \dots + P^{k_t}|\beta_t|)^{-1/k_1}. \tag{2.4}$$

Now let $W \le R$ be a parameter at our disposal. We define the pruned major arcs $\mathfrak{N}$ to be the union of the sets

$$\mathfrak{N}(q,\mathbf{a}) = \{\boldsymbol{\alpha} \in \mathbb{T}^t : |\alpha_i - a_i/q| < WP^{-k_i} \quad (i = 1, \dots, t)\} \tag{2.5}$$

with $0 \le a_1, \dots, a_t \le q \le W$ and $(q, a_1, \dots, a_t) = 1$. Note here that the condition $(q, a_1, \dots, a_{t-1}) \le P^{\varepsilon}$ is automatically satisfied, since $q \le R$ and R is a sufficiently small power of P. Furthermore, the $\mathfrak{N}(q,\mathbf{a})$ are pairwise disjoint, even when $k_t = 1$. We need the following easy extension of Wooley [19], Lemma 9.2 (see also [14], Lemma 7.2), in order to deal with the major arcs.

Lemma 2.1. *If T is a real number with $T > k_1(t+1)$, then one has*

$$\int_{\mathfrak{M}} |F(\boldsymbol{\alpha})|^T\, d\boldsymbol{\alpha} \ll P^{T-K}$$

and, for some $\sigma > 0$,

$$\int_{\mathfrak{M}\setminus\mathfrak{N}} |F(\boldsymbol{\alpha})|^T\, d\boldsymbol{\alpha} \ll P^{T-K} W^{-\sigma}.$$

Proof. Suppose that $T > k_1(t+1)$. When $\boldsymbol{\alpha} \in \mathfrak{M}(q,\mathbf{a}) \subseteq \mathfrak{M}$, we write $\beta_i = \alpha_i - a_i/q$ for $i = 1, \dots, t$, and define the function $V(\boldsymbol{\alpha}) = q^{-1}S(q,\mathbf{a})v(\boldsymbol{\beta})$. In order to make this well-defined when $k_t = 1$, we can associate $\boldsymbol{\alpha}$ to the $\mathfrak{M}(q,\mathbf{a})$ having minimal q, since the arcs corresponding to a fixed q are pairwise disjoint. Then when $\boldsymbol{\alpha} \in \mathfrak{M}(q,\mathbf{a}) \subseteq \mathfrak{M}$, we have by Lemma 4.4 of Baker [2] that

$$|F(\boldsymbol{\alpha})|^T \ll |V(\boldsymbol{\alpha})|^T + P^\varepsilon(q^{1-1/k_1})^T.$$

Write $\mathfrak{W}$ for either $\mathfrak{M}$ or $\mathfrak{M}\backslash\mathfrak{N}$. Then one sees easily from (2.1) that

$$\int_{\mathfrak{W}} P^\varepsilon(q^{1-1/k_1})^T\, d\boldsymbol{\alpha} \ll P^{t-K+\varepsilon}\sum_{q\le P} q^{T(1-1/k_1)} \ll P^{T-K-\delta}$$

for some $\delta > 0$, since $T > k_1(t+1)$. Now by (2.2), one has

$$\int_{\mathfrak{W}} |V(\boldsymbol{\alpha})|^T\, d\boldsymbol{\alpha} \ll \sum_{q\le P}\sum_{\mathbf{a}\in[0,q]^t} q^{-T/k_1+\varepsilon}\int_{\mathfrak{W}(q,\mathbf{a})} |v(\boldsymbol{\beta})|^T\, d\boldsymbol{\alpha}, \tag{2.6}$$

where have written $\mathfrak{W}(q,\mathbf{a}) = \mathfrak{M}(q,\mathbf{a})$ when $\mathfrak{W} = \mathfrak{M}$ and $\mathfrak{W}(q,\mathbf{a}) = \mathfrak{M}(q,\mathbf{a})\backslash\mathfrak{N}(q,\mathbf{a})$ when $\mathfrak{W} = \mathfrak{M}\backslash\mathfrak{N}$. Now set $Y = 1$ if $\mathfrak{W} = \mathfrak{M}$ or if $q > W$, and put $Y = W$ otherwise. Then by applying (2.3) and making a change of variable, one finds that

$$\begin{aligned}\int_{\mathfrak{W}(q,\mathbf{a})} |v(\boldsymbol{\beta})|^T\, d\boldsymbol{\alpha} &\ll P^T\int_{\mathfrak{W}(q,\mathbf{a})}\prod_{i=1}^t (1+P^{k_i}|\beta_i|)^{-T/tk_1}\, d\boldsymbol{\alpha}\\ &\ll P^{T-K}Y^{1-T/tk_1}.\end{aligned}$$

Thus on writing $Z = 1$ if $\mathfrak{W} = \mathfrak{M}$ and $Z = W$ if $\mathfrak{W} = \mathfrak{M}\backslash\mathfrak{N}$, we obtain from (2.6) that

$$\begin{aligned}\int_{\mathfrak{W}} |V(\boldsymbol{\alpha})|^T\, d\boldsymbol{\alpha} &\ll P^{T-K}\left(\sum_{q\le W} Z^{1-T/tk_1}q^{t-T/k_1+\varepsilon} + \sum_{q>W} q^{t-T/k_1+\varepsilon}\right)\\ &\ll P^{T-K}\left(Z^{1-T/tk_1} + W^{t+1-T/k_1+\varepsilon}\right),\end{aligned}$$

and the lemma follows. $\square$

Actually, only the first estimate of Lemma 2.1 will be required for our purposes, as the argument of §3 completely avoids pruning. We have included the second estimate since it may be useful in other contexts. We are now able to establish mean value estimates in which one obtains the full savings of P^K over the trivial bound.

Lemma 2.2. *Let u, Δ_u, and $\sigma(\mathbf{k})$ be as in the statement of Theorem* 1.1*, and suppose that ν is an even integer satisfying $\nu\sigma(\mathbf{k}) > \Delta_u$ and $2u + \nu > k_1(t+1)$. Then one has*

$$\int_{\mathbb{T}^t} |f(\boldsymbol{\alpha})|^{2u+\nu}\, d\boldsymbol{\alpha} \ll P^{2u+\nu-K}.$$

Proof. Write $\mathcal{I}$ for the integral in question. Since ν is even, one sees by considering the underlying diophantine equations that

$$\mathcal{I} \le \int_{\mathbb{T}^t} |F(\boldsymbol{\alpha})^\nu f(\boldsymbol{\alpha})^{2u}|\, d\boldsymbol{\alpha},$$

and we now dissect into major and minor arcs as in (2.1). Since $\nu\sigma(\mathbf{k}) > \Delta_u$, we have

$$\int_{\mathfrak{m}} |F(\boldsymbol{\alpha})^\nu f(\boldsymbol{\alpha})^{2u}|\, d\boldsymbol{\alpha} \ll P^{2u+\nu-K-\delta}$$

for some $\delta > 0$. We may therefore suppose that the contribution from the major arcs dominates, in which case we have by Hölder's inequality that

$$\mathcal{I} \ll \int_{\mathfrak{M}} |F(\boldsymbol{\alpha})^\nu f(\boldsymbol{\alpha})^{2u}|\, d\boldsymbol{\alpha} \ll \left(\int_{\mathfrak{M}} |F(\boldsymbol{\alpha})|^{2u+\nu}\, d\boldsymbol{\alpha} \right)^{\nu/(2u+\nu)} \mathcal{I}^{2u/(2u+\nu)},$$

and the result now follows from Lemma 2.1. □

Recall the definition of $F_i(\boldsymbol{\alpha})$ from §1, and define $f_i(\boldsymbol{\alpha})$ similarly.

Lemma 2.3. *Let u, ν, Δ_u, and $\sigma(\mathbf{k})$ be as in the statement of Lemma 2.2, and suppose that ω is a real number satisfying $\omega\sigma(\mathbf{k}) > \Delta_u$ and $\omega(2u+\nu) > \nu k_1(t+1)$. Then for any i and j, one has*

$$\int_{\mathbb{T}^t} |F_i(\boldsymbol{\alpha})^\omega f_j(\boldsymbol{\alpha})^{2u}|\, d\boldsymbol{\alpha} \ll P^{2u+\omega-K}.$$

Proof. We again dissect into major and minor arcs. Write $\mathfrak{M}_i$ for the set of $\boldsymbol{\alpha} \in \mathbb{T}^t$ for which $\boldsymbol{\lambda}_i\boldsymbol{\alpha} \in \mathfrak{M}$, where $\mathfrak{M}$ is as in (2.1), and write $\mathfrak{m}_i = \mathbb{T}^t \setminus \mathfrak{M}_i$. Since $\omega\sigma(\mathbf{k}) > \Delta_u$, we find after a change of variable that

$$\int_{\mathfrak{m}_i} |F_i(\boldsymbol{\alpha})^\omega f_j(\boldsymbol{\alpha})^{2u}|\, d\boldsymbol{\alpha} \ll P^{2u+\omega-K-\delta}$$

for some $\delta > 0$. For brevity, let us write $s = 2u+\nu$. Then we have by Hölder's inequality that

$$\int_{\mathfrak{M}_i} |F_i(\boldsymbol{\alpha})^\omega f_j(\boldsymbol{\alpha})^{2u}|\, d\boldsymbol{\alpha} \ll \left(\int_{\mathfrak{M}_i} |F_i(\boldsymbol{\alpha})|^{\omega s/\nu}\, d\boldsymbol{\alpha} \right)^{\nu/s} \left(\int_{\mathbb{T}^t} |f_j(\boldsymbol{\alpha})|^s\, d\boldsymbol{\alpha} \right)^{2u/s}.$$

In view of the condition $\omega(2u+\nu) > \nu k_1(t+1)$, the desired estimate follows from Lemmas 2.1 and 2.2 on making a change of variable. □

Next we need a Weyl-type lemma similar to Theorem 5.1 of Baker [2], but with certain factors of P^ε removed from the statement, as in Lemma 6 of Freeman [11]. Here we establish a version of Freeman's lemma through a simple argument suggested recently by Professor Wooley, who has graciously allowed us to reproduce it here.

Lemma 2.4. *Suppose that $|F(\boldsymbol{\alpha})| \geq PA^{-1}$, where $A \leq P^{2^{1-k_1}-\eta}$ for some $\eta > 0$, and that P is sufficiently large in terms of η and $\mathbf{k}$. Then there exists a positive integer q and integers $a_1, \dots, a_t$ satisfying $(q, a_1, \dots, a_t) = 1$ such that*

$$q \ll A^{2k_1} \quad \textit{and} \quad |q\alpha_i - a_i| \ll A^{2k_1} P^{-k_i} \quad (1 \leq i \leq t),$$

where the implicit constants depend at most on η and $\mathbf{k}$.

Proof. Clearly, we may suppose throughout that η is sufficiently small. We may further suppose that $\boldsymbol{\alpha} \in \mathbb{T}^t$, since the result then extends to all $\boldsymbol{\alpha}$ by adding a suitable multiple of q to each a_i. We define the major arcs $\mathfrak{M}(q, \mathbf{a})$ as in (2.1) and introduce the function $V_1(\boldsymbol{\alpha})$, defined by

$$V_1(\boldsymbol{\alpha}) = q^{-1} S(q, \mathbf{a}) v(\boldsymbol{\alpha} - \mathbf{a}/q) + P^\eta q^{1-1/k_1}$$

when $\boldsymbol{\alpha} \in \mathfrak{M}(q, \mathbf{a}) \subseteq \mathfrak{M}$ and by $V_1(\boldsymbol{\alpha}) = 0$ otherwise. We adopt the same convention as in the proof of Lemma 2.1 to ensure that $V_1(\boldsymbol{\alpha})$ is well-defined when $k_t = 1$. Now by combining Lemma 4.4 and Theorem 5.1 of Baker [2], we find that

$$|F(\boldsymbol{\alpha})| \ll P^{1-2^{1-k_1}+\eta/2} + |V_1(\boldsymbol{\alpha})|$$

for all $\boldsymbol{\alpha} \in \mathbb{T}^t$. Furthermore, it follows from (2.2) and (2.3) that for each $\boldsymbol{\alpha}$ there is a positive integer q and integers $a_1, \ldots, a_t$ such that

$$|V_1(\boldsymbol{\alpha})| \ll Pq^{\eta}(q + |q\alpha_1 - a_1|P^{k_1} + \cdots + |q\alpha_t - a_t|P^{k_t})^{-1/k_1}.$$

On recalling the hypothesis of the statement of the lemma, we conclude that

$$q^{-\eta k_1}(q + |q\alpha_1 - a_1|P^{k_1} + \cdots + |q\alpha_t - a_t|P^{k_t}) \ll A^{k_1},$$

and the result now follows easily on taking $\eta \leq 1/2k_1$. □

We are now in a position to obtain a weak minor arc estimate of the type described in the introduction. The argument given below is essentially due to Freeman [10].

Lemma 2.5. *Fix i with $1 \leq i \leq t$ for which $\lambda_{i1}/\lambda_{i2}$ is irrational, and fix $\delta > 0$. There is a positive, real-valued function $T_i(P)$ such that $T_i(P) \to \infty$ as $P \to \infty$ and*

$$\lim_{P\to\infty} \left(\sup_{\boldsymbol{\alpha}\in\mathfrak{n}_i(P)} \frac{|F_1(\boldsymbol{\alpha}; P)F_2(\boldsymbol{\alpha}; P)|}{P^2} \right) = 0,$$

where

$$\mathfrak{n}_i(P) = \{\boldsymbol{\alpha} \in \mathbb{R}^t : (\log P)^{\delta} P^{-k_i} \leq |\alpha_i| \leq T_i(P)\}.$$

Proof. We first show that the result holds when $\mathfrak{n}_i(P)$ is replaced by the set

$$\widetilde{\mathfrak{n}}_i(P) = \{\boldsymbol{\alpha} \in \mathbb{R}^t : (\log P)^{\delta} P^{-k_i} \leq |\alpha_i| \leq T\}$$

for any fixed real number $T \geq 1$. If $F_1(\boldsymbol{\alpha}; P)F_2(\boldsymbol{\alpha}; P)$ is not $o(P^2)$ on $\widetilde{\mathfrak{n}}_i(P)$, then we can find $\varepsilon > 0$, a sequence of positive real numbers $\{P_n\}$ tending to ∞, and a sequence of vectors $\{\boldsymbol{\alpha}_n\}$ with $\boldsymbol{\alpha}_n = (\alpha_{n1}, \ldots, \alpha_{nt}) \in \widetilde{\mathfrak{n}}_i(P_n)$, having the property that

$$|F_1(\boldsymbol{\alpha}_n; P_n)F_2(\boldsymbol{\alpha}_n; P_n)| \geq \varepsilon P_n^2$$

for all positive integers n. On making a trivial estimate, it follows that for each n one has

$$|F_j(\boldsymbol{\alpha}_n; P_n)| \geq \varepsilon P_n \qquad (j = 1, 2).$$

Whenever n is large enough so that $P_n \geq \varepsilon^{-2^{k_1}}$, we may apply Lemma 2.4 with $A = 1/\varepsilon$ to obtain integers q_{nj} and a_{nj} satisfying

$$q_{nj} \ll \varepsilon^{-2k_1} \quad \text{and} \quad |\lambda_{ij}\alpha_{ni}q_{nj} - a_{nj}| \ll \varepsilon^{-2k_1}P_n^{-k_i} \qquad (j = 1, 2), \tag{2.7}$$

where the implicit constants depend only on $\mathbf{k}$. It follows that, for n sufficiently large, one has

$$a_{nj} \ll |\lambda_{ij}|T\varepsilon^{-2k_1} + \varepsilon^{-2k_1}P_n^{-k_i} \ll 1,$$

and hence there are only finitely many possible 4-tuples $(a_{n1}, q_{n1}, a_{n2}, q_{n2})$. So there must be a 4-tuple (a_1, q_1, a_2, q_2) that occurs for infinitely many n, and we let $\mathcal{S}$ denote this sequence of values of n. Then when $n \in \mathcal{S}$, we have by (2.7) that

$$q_j \ll 1 \quad \text{and} \quad \lambda_{ij}\alpha_{ni} = \frac{a_j}{q_j} + O(P_n^{-k_i}) \qquad (j = 1, 2), \tag{2.8}$$

where the implicit constants depend on ε, $\mathbf{k}$ and $\boldsymbol{\lambda}_i$. Since each α_{ni} lies in the compact set $[-T,T]$, we can find a subsequence $\mathcal{S}'\subseteq\mathcal{S}$ such that $\{\alpha_{ni}\}$ converges to a limit α_i^* as $n\to\infty$ through $\mathcal{S}'$.

If $\alpha_i^*=0$, then one has $|\alpha_{ni}|<|2\lambda_{ij}q_j|^{-1}$ for all sufficiently large $n\in\mathcal{S}'$. We therefore deduce from (2.8) that $a_1=a_2=0$, and hence that $\alpha_{ni}\ll P_n^{-k_i}$ for large n. But this contradicts the fact that $\boldsymbol{\alpha}_n\in\widetilde{\mathfrak{n}}_i(P_n)$, so we are forced to conclude that $\alpha_i^*\neq 0$.

Thus after letting $n\to\infty$ through the elements of $\mathcal{S}'$ and dividing the two equations in (2.8), we find that $\lambda_{i1}/\lambda_{i2}=a_1q_2/(a_2q_1)$, contradicting the assumption that $\lambda_{i1}/\lambda_{i2}$ is irrational. We therefore conclude that $F_1(\boldsymbol{\alpha};P)F_2(\boldsymbol{\alpha};P)=o(P^2)$ on $\widetilde{\mathfrak{n}}_i(P)$. In particular, for each positive integer m, there is a real number P_m such that

$$\frac{|F_1(\boldsymbol{\alpha};P)F_2(\boldsymbol{\alpha};P)|}{P^2}\le\frac{1}{m}\quad\text{whenever}\quad P\ge P_m\quad\text{and}\quad(\log P)^\delta P^{-k_i}\le|\alpha_i|\le m,$$

and we may clearly suppose that the sequence $\{P_m\}$ is non-decreasing and tends to infinity. To complete the proof of the lemma, it now suffices to define $T_i(P)=m$ whenever $P_m\le P<P_{m+1}$. □

3. The Circle Method for Mixed Systems

Since we have imposed no irrationality assumption concerning the coefficient ratios in (1.1), it may be the case that some of the forms have all their coefficients in rational ratio and hence are multiples of integral forms. For a sufficiently small choice of τ, the corresponding inequalities in (1.1) are equivalent to homogeneous equations with integral coefficients. In the remaining forms, we may further reduce to the case $\tau=1$ by replacing λ_{ij} by λ_{ij}/τ throughout. Thus we see that solving (1.1) amounts to solving a system of the shape

$$\begin{aligned} c_{i1}x_1^{m_i}+\cdots+c_{is}x_s^{m_i}&=0\quad(1\le i\le r)\\ |\lambda_{i1}x_1^{m_i}+\cdots+\lambda_{is}x_s^{m_i}|&<1\quad(r<i\le t),\end{aligned}\tag{3.1}$$

where $c_{ij}\in\mathbb{Z}$, $\lambda_{ij}\in\mathbb{R}$, and $i\neq j\Rightarrow m_i\neq m_j$. Furthermore, for each i with $r<i\le t$, one has $\lambda_{ij}/\lambda_{ik}\notin\mathbb{Q}$ for some j and k. Since we have assumed all coefficients to be non-zero, it is easy to show, after a rearrangement of variables, that for each such i we may take the corresponding j and k to satisfy $1\le j,k\le\max(2,t-r)$. Plainly, we may further suppose that

$$m_1>\cdots>m_r\quad\text{and}\quad m_{r+1}>\cdots>m_t.$$

We also write $k_1=\max(m_1,m_{r+1})\ge 2$ and $K=m_1+\cdots+m_t$ to coincide with the notation of the previous sections. Note that we may of course have all inequalities ($r=0$) or all equations ($r=t$). When $1\le r<t$, the system (3.1) is genuinely mixed, and we require a hybrid of the usual Hardy-Littlewood and Davenport-Heilbronn methods to analyze it.

We introduce the functions

$$K(\alpha)=\left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2\quad\text{and}\quad\mathcal{K}(\boldsymbol{\alpha})=\prod_{i=r+1}^{t}K(\alpha_i),\tag{3.2}$$

and observe that one has

$$K(\alpha)\ll\min(1,|\alpha|^{-2}).\tag{3.3}$$

It is not difficult to show (see for example Baker [2], Lemma 14.1) that the Fourier transform of K satisfies

$$\widehat{K}(y) = \int_{-\infty}^{\infty} K(\alpha)\, e(\alpha y)\, d\alpha = \max(0, 1 - |y|) \tag{3.4}$$

for all real numbers y.

Now write $s = 2u + v$, where u and v satisfy the hypotheses of Theorem 1.1, and let $N(P)$ denote the number of solutions of the system (3.1) satisfying

$$1 \le x_j \le P \quad (j = 1, \dots, v) \quad \text{and} \quad x_j \in \mathcal{A}(P, R) \quad (j = v+1, \dots, s).$$

On writing

$$\mathcal{F}(\boldsymbol{\alpha}) = \prod_{j=1}^{v} F_j(\boldsymbol{\alpha}) \prod_{j=v+1}^{s} f_j(\boldsymbol{\alpha})$$

and using (3.4), one finds that

$$N(P) \ge \int_{\mathfrak{U}} \mathcal{F}(\boldsymbol{\alpha})\, \mathcal{K}(\boldsymbol{\alpha})\, d\boldsymbol{\alpha}, \tag{3.5}$$

where $\mathfrak{U} = \mathbb{T}^r \times \mathbb{R}^{t-r}$. In order to obtain a lower bound for this integral, we consider the following dissection of $\mathfrak{U}$. Let $W = (\log P)^{1/6t}$, and write

$$\mathfrak{N}(q, \mathbf{a}) = \{\boldsymbol{\alpha} \in \mathbb{T}^t : |\alpha_i - a_i/q| < W P^{-m_i} \quad (1 \le i \le t)\}. \tag{3.6}$$

We define the major arcs $\mathfrak{N}$ to be the union of the sets $\mathfrak{N}(q, a_1, \dots, a_r, 0, \dots, 0)$ with $0 \le a_1, \dots, a_r \le q \le W$ and $(q, a_1, \dots, a_r) = 1$. For $r + 1 \le i \le t$, let $T_i(P)$ be as in Lemma 2.5, but of course re-indexed as in (3.1), and define the trivial arcs to be the set

$$\mathfrak{t} = \bigcup_{i=r+1}^{t} \{\boldsymbol{\alpha} \in \mathfrak{U} : |\alpha_i| > T_i(P)\}.$$

Finally, we define the minor arcs by

$$\mathfrak{n} = \mathfrak{U} \setminus (\mathfrak{N} \cup \mathfrak{t}).$$

By expressing the trivial arcs as a union of t-dimensional unit hypercubes, recalling (3.3), and applying Lemma 2.3 with $\omega = v$ and $\nu = 2\lceil v/2 \rceil$, we find that under the hypotheses of Theorem 1.1 one has

$$\int_{\mathfrak{t}} \mathcal{F}(\boldsymbol{\alpha})\, \mathcal{K}(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} \ll P^{s-K} \sum_{i=r+1}^{t} T_i(P)^{-1} = o(P^{s-K}). \tag{3.7}$$

Most of the work required to handle the minor arcs was accomplished in Lemma 2.5. Before proceeding with the analysis, we pause to record an easy consequence of that lemma.

Lemma 3.1. *Write* $w = \max(2, t - r)$. *Then one has*

$$\sup_{\boldsymbol{\alpha} \in \mathfrak{n}} \prod_{j=1}^{w} |F_j(\boldsymbol{\alpha})| = o(P^w).$$

Proof. For each i with $r+1 \le i \le t$, the discussion following (3.1) shows that we can find indices $j = j(i)$ and $k = k(i)$, with $1 \le j, k \le w$, for which $\lambda_{ij}/\lambda_{ik}$ is irrational. Thus on writing

$$\mathfrak{n}_i = \{\boldsymbol{\alpha} \in \mathfrak{n} : |\alpha_i| \ge WP^{-k_i}\},$$

we deduce from an obvious generalization of Lemma 2.5 that

$$\sup_{\boldsymbol{\alpha}\in\mathfrak{n}_i} |F_j(\boldsymbol{\alpha})F_k(\boldsymbol{\alpha})| = o(P^2) \qquad (r+1 \le i \le t).$$

Now let $\mathfrak{n}^* = \mathfrak{n} \setminus (\mathfrak{n}_{r+1} \cup \cdots \cup \mathfrak{n}_t)$. Then by combining Lemma 2.4 with the argument of Wooley [19], Lemma 7.4, one sees that

$$\sup_{\boldsymbol{\alpha}\in\mathfrak{n}^*} |F_j(\boldsymbol{\alpha})| = o(P)$$

for each j, and the lemma follows. □

We can now rapidly dispose of the minor arcs. First of all, by using (3.3), we see that for any $\delta > 0$ one has

$$\int_{\mathfrak{n}} \mathcal{F}(\boldsymbol{\alpha})\,\mathcal{K}(\boldsymbol{\alpha})\,d\boldsymbol{\alpha} \ll \left(\sup_{\boldsymbol{\alpha}\in\mathfrak{n}} \prod_{i=1}^{v} |F_i(\boldsymbol{\alpha})|\right)^{\delta} \int_{\mathbb{T}^t} |F_j(\boldsymbol{\alpha})|^{v(1-\delta)} |f_k(\boldsymbol{\alpha})|^{2u}\,d\boldsymbol{\alpha}$$

for some j and k with $1 \le j \le v$ and $v+1 \le k \le s$. Now set $\omega = v(1-\delta)$, and let ν be the smallest even integer with $\nu \ge v$. In view of the hypotheses of Theorem 1.1, we have $\omega\sigma(\mathbf{k}) > \Delta_u$ and $\omega(2u+\nu) > \nu k_1(t+1)$, on choosing δ sufficiently small. We also have $v \ge w$, and it therefore follows from Lemma 2.3 and Lemma 3.1 that

$$\int_{\mathfrak{n}} \mathcal{F}(\boldsymbol{\alpha})\,\mathcal{K}(\boldsymbol{\alpha})\,d\boldsymbol{\alpha} = o(P^{s-K}). \tag{3.8}$$

It now suffices to obtain a lower bound for the contribution of the major arcs $\mathfrak{N}$. Write $\boldsymbol{\theta}_j = (c_{1j}, \dots, c_{rj}, \lambda_{(r+1)j}, \dots, \lambda_{tj})$, and let $S_j(q, \mathbf{a}) = S(q, \boldsymbol{\theta}_j\mathbf{a})$, $v_j(\boldsymbol{\beta}) = v(\boldsymbol{\theta}_j\boldsymbol{\beta})$, and $w_j(\boldsymbol{\beta}) = w(\boldsymbol{\theta}_j\boldsymbol{\beta})$. Then when $\boldsymbol{\alpha} \in \mathfrak{N}(q, \mathbf{a}) \subseteq \mathfrak{N}$, we have by Theorem 7.2 of Vaughan [18] that

$$F_j(\boldsymbol{\alpha}) = q^{-1}S_j(q, \mathbf{a})v_j(\boldsymbol{\beta}) + O(W^2)$$

and by Lemma 8.5 of Wooley [19] that

$$f_j(\boldsymbol{\alpha}) = q^{-1}S_j(q, \mathbf{a})w_j(\boldsymbol{\beta}) + O(W^2P(\log P)^{-1}).$$

Since $\operatorname{meas}(\mathfrak{N}) \ll W^{r+t+1}P^{-K}$, it follows easily that

$$\int_{\mathfrak{N}} \mathcal{F}(\boldsymbol{\alpha})\,\mathcal{K}(\boldsymbol{\alpha})\,d\boldsymbol{\alpha} = \mathfrak{S}(W)J(W) + O(P^{s-K}W^{r+t+3}(\log P)^{-1}), \tag{3.9}$$

where

$$\mathfrak{S}(W) = \sum_{q\le W} S(q), \qquad S(q) = \sum_{\substack{1\le a_1,\dots,a_r\le q\\ (q,a_1,\dots,a_r)=1}} \prod_{j=1}^{s} q^{-1}S_j(q, \mathbf{a}),$$

and

$$J(W) = \int_{-WP^{-m_1}}^{WP^{-m_1}} \cdots \int_{-WP^{-m_t}}^{WP^{-m_t}} \mathcal{K}(\boldsymbol{\beta}) \prod_{j=1}^{v} v_j(\boldsymbol{\beta}) \prod_{j=v+1}^{s} w_j(\boldsymbol{\beta})\,d\boldsymbol{\beta}.$$

We now complete the truncated singular series and singular integral to infinity by writing

$$\mathfrak{S} = \sum_{q=1}^{\infty} S(q) \qquad \text{and} \qquad J = \int_{\mathbb{R}^t} \mathcal{K}(\boldsymbol{\beta}) \prod_{j=1}^{v} v_j(\boldsymbol{\beta}) \prod_{j=v+1}^{s} w_j(\boldsymbol{\beta})\, d\boldsymbol{\beta}.$$

Lemma 3.2. *Whenever $s > k_1(t+1)$, the series $\mathfrak{S}$ and the integral J are absolutely convergent, and moreover one has*

$$\mathfrak{S} - \mathfrak{S}(W) \ll W^{-\sigma} \quad \textit{and} \quad J - J(W) \ll P^{s-K} W^{-\sigma}$$

for some $\sigma > 0$.

Proof. On recalling (2.2), we find that

$$S(q) \ll q^{r-s/m_1+\varepsilon}, \tag{3.10}$$

and the bounds for $\mathfrak{S}$ follow immediately whenever $s > m_1(r+1)$. Now by (2.3), (2.4), (3.3), and a change of variables, we have

$$J \ll P^{s-K} \int_{\mathbb{R}^t} \prod_{i=1}^{t} (1+|\beta_i|)^{-s/tk_1}\, d\boldsymbol{\beta} \ll P^{s-K}$$

whenever $s > tk_1$, and the bound for $J - J(W)$ follows similarly. □

It follows immediately from (3.9), Lemma 3.2, and our definition of W that

$$\int_{\mathfrak{N}} \mathcal{F}(\boldsymbol{\alpha})\, \mathcal{K}(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} = \mathfrak{S}J + O(P^{s-K}(\log P)^{-\sigma}) \tag{3.11}$$

for some $\sigma > 0$, so it suffices to analyze $\mathfrak{S}$ and J. In view of our assumptions concerning the integral sub-system in (1.1), the singular series is easily handled by the methods of Wooley [19]. For the singular integral, however, we follow the approach of Schmidt [17], which avoids the use of Fourier's Integral Theorem. We record our results concerning these two objects in the following lemma.

Lemma 3.3. *Whenever $s > k_1(t+1)$, one has $\mathfrak{S} > 0$ and $J \gg P^{s-K}$.*

Proof. We first deal with the singular series. On recalling (3.10), we see that the series

$$\varpi_p = \sum_{h=0}^{\infty} S(p^h)$$

is absolutely convergent and satisfies $\varpi_p - 1 \ll p^{-1-\delta}$ for some $\delta > 0$ whenever $s > m_1(r+1)$. We therefore find as in Wooley [19], Lemma 10.8, that $\mathfrak{S}$ is represented by the absolutely convergent product $\mathfrak{S} = \prod_p \varpi_p$, and that there exists an integer p_0 such that

$$\frac{1}{2} \le \prod_{p \ge p_0} \varpi_p \le \frac{3}{2}.$$

It therefore suffices to show that $\varpi_p > 0$ for primes $p < p_0$. Let $M_s(q)$ denote the number of solutions of the system of congruences

$$c_{i1}x_1^{m_i} + \cdots + c_{is}x_s^{m_i} \equiv 0 \pmod{q} \quad (1 \le i \le r).$$

By applying the argument of [18], Lemma 2.12, as in [19], Lemma 10.9, we find that

$$\sum_{d|q} S(d) = q^{r-s} M_s(q),$$

and it follows that

$$\varpi_p = \lim_{h\to\infty} \sum_{d|p^h} S(d) = \lim_{h\to\infty} p^{h(r-s)} M_s(p^h).$$

Since we have assumed that the integral sub-system consisting of the r equations in (3.1) possesses a non-singular p-adic solution for each prime p, we may apply a Hensel's Lemma argument as in Wooley [19], Lemma 6.7, to conclude that there exists an integer $u = u(p) < \infty$ such that for all $h \geq u$ one has

$$M_s(p^h) \geq p^{(h-u)(s-r)}.$$

It follows that $\varpi_p \geq p^{u(r-s)}$ for each $p < p_0$, and thus $\mathfrak{S} > 0$.

It remains to handle the singular integral. Let T be a positive real number, and introduce the functions

$$K_T(\beta) = \left(\frac{\sin \pi\beta T^{-1}}{\pi\beta T^{-1}}\right)^2 \quad \text{and} \quad \mathcal{K}_T(\boldsymbol{\beta}) = \prod_{i=1}^{r} K_T(\beta_i).$$

Then on recalling (3.2) and (3.4), we find that

$$\widehat{K}_T(y) = \int_{-\infty}^{\infty} K_T(\beta)\, e(\beta y)\, d\beta = T \max(0, 1 - T|y|) \tag{3.12}$$

for all real numbers y. Further, write

$$J_T = \int_{\mathbb{R}^t} \mathcal{K}_T(\boldsymbol{\beta})\ \mathcal{K}(\boldsymbol{\beta})\ \prod_{j=1}^{v} v_j(\boldsymbol{\beta}) \prod_{j=v+1}^{s} w_j(\boldsymbol{\beta})\ d\boldsymbol{\beta}.$$

It follows from (2.3), (2.4), and (3.3) that

$$J - J_T \ll P^s \int_{\mathbb{R}^t} (1 - \mathcal{K}_T(\boldsymbol{\beta})) \prod_{i=1}^{t} (1 + P^{k_i}|\beta_i|)^{-s/tk_1}\, d\boldsymbol{\beta}, \tag{3.13}$$

and a simple calculation reveals that

$$1 - \mathcal{K}_T(\boldsymbol{\beta}) \ll \min(1, |\boldsymbol{\beta}|^2 T^{-2}).$$

Thus on making a change of variables in (3.13) and considering the resulting integral over the regions $|\boldsymbol{\beta}| \leq T$ and $|\boldsymbol{\beta}| > T$ separately, it is easily shown that

$$J - J_T \ll P^{s-K} T^{-1/t},$$

whenever $s > k_1(t+1)$. Hence for any fixed P, we have

$$J = \lim_{T\to\infty} J_T, \tag{3.14}$$

and so it suffices to analyze J_T. By making a change of variable, we find that

$$J_T = P^s \int_{\mathfrak{B}} \mathcal{H}(\boldsymbol{\gamma}) \prod_{i=1}^{r} \widehat{K}_T(P^{m_i} g_i(\boldsymbol{\gamma})) \prod_{i=r+1}^{t} \widehat{K}(P^{m_i} g_i(\boldsymbol{\gamma}))\, d\boldsymbol{\gamma}, \tag{3.15}$$

where we have written

$$g_i(\boldsymbol{\gamma}) = \begin{cases} c_{i1}\gamma_1^{m_i} + \cdots + c_{is}\gamma_s^{m_i} & (1 \le i \le r) \\ \lambda_{i1}\gamma_1^{m_i} + \cdots + \lambda_{is}\gamma_s^{m_i} & (r < i \le t), \end{cases}$$

$$\mathcal{H}(\boldsymbol{\gamma}) = \prod_{j=v+1}^{s} \rho\left(\frac{\log(P\gamma_j)}{\log R}\right), \qquad \text{and} \qquad \mathfrak{B} = [0,1]^v \times [R/P, 1]^{2u}.$$

Since we have assumed that the system $g_1(\boldsymbol{\gamma}) = \cdots = g_t(\boldsymbol{\gamma}) = 0$ possesses a non-singular real solution $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_s)$, the Implicit Function Theorem ensures that locally near $\boldsymbol{\eta}$ there is an $(s-t)$-dimensional space of real solutions, continuously parameterized by $s-t$ of the coordinates. Therefore, by exploiting continuity as in the proof of [19], Lemma 6.2, we may suppose that each η_j is non-zero. Further, by replacing x_j by $-x_j$ and changing the signs of the corresponding coefficients if necessary, we may suppose that each η_j is positive and hence that $\boldsymbol{\eta}$ lies in the interior of $\mathfrak{B}$ for P sufficiently large. Now consider the map $\varphi : \mathbb{R}^s \to \mathbb{R}^s$ defined by

$$\varphi_j(\boldsymbol{\gamma}) = g_j(\boldsymbol{\gamma}) \quad (1 \le j \le t) \quad \text{and} \quad \varphi_j(\boldsymbol{\gamma}) = \gamma_j \quad (t < j \le s).$$

By the Inverse Function Theorem, there is an open set $U \subseteq \mathfrak{B}$ containing $\boldsymbol{\eta}$, and an open set V containing $(0, \ldots, 0, \eta_{t+1}, \ldots, \eta_s)$, such that φ maps U injectively onto V. Since $\mathcal{H}(\boldsymbol{\gamma}) \gg 1$ on $\mathfrak{B}$ and the integrand in (3.15) is non-negative, we have by a change of variable that

$$J_T \gg P^{s-K} \int_{V^*} \prod_{i=1}^{r} \widehat{K}_T(u_i) \prod_{i=r+1}^{t} \widehat{K}(u_i)\, d\mathbf{u}, \tag{3.16}$$

where V^* is obtained by projecting V onto the first t components and then stretching by a factor of P^{m_i} in the direction of u_i. In particular, it is clear that V^* contains the set

$$\mathfrak{D} = \left[-\frac{1}{2T}, \frac{1}{2T}\right]^r \times \left[-\frac{1}{2}, \frac{1}{2}\right]^{t-r}$$

whenever $T \ge 1$ and P is sufficiently large. By (3.4) and (3.12), the integrand in (3.16) is bounded below on $\mathfrak{D}$ by $2^{-t}T^r$, and one also has $\operatorname{meas}(\mathfrak{D}) \gg T^{-r}$. It follows immediately that $J_T \gg P^{s-K}$ for $T \ge 1$, where the implicit constant is independent of T. The lemma therefore follows from (3.14) on letting $T \to \infty$. $\square$

In view of Lemma 3.3, the proof of Theorem 1.1 is now completed on assembling (3.7), (3.8), and (3.11). Corollary 1.2 follows immediately on comparing the parameters in Table 7.1 of [14] with the conditions of Theorem 1.1.

We now indicate how to deduce Corollary 1.3. Note first of all that the corollary is well-known when $t = 1$, so we may suppose that $t \ge 2$ if necessary. By applying Theorems 2 and 3 of Wooley [20], we find that a mean value estimate of the shape (1.5) holds with

$$\Delta_{\tilde{u}} \ll (t \log k_1)^{-1} \tag{3.17}$$

and $\tilde{u} = \min(u_1, u_2)$, where

$$u_1 \sim \tfrac{1}{2} t k_1 \left(\log k_1 + 3\log t + 4 \log\log k_1\right)$$

and

$$u_2 \sim \tfrac{1}{2} k_1 \left(\log(k_1 \cdots k_t) + 3t^2 + 6t \log\log k_1 + 2t \log t\right).$$

Thus by taking $\lambda = 1/2t$ in Theorem 4 of Wooley [20], one obtains the Weyl estimate

$$\sup_{\boldsymbol{\alpha}\in\mathfrak{m}_{1/2}} |f(\boldsymbol{\alpha})| \ll P^{1-\sigma(\mathbf{k})+\varepsilon},$$

where $\sigma(\mathbf{k})^{-1} \sim 4t\tilde{u} \ll t^2 k_1 \log k_1$, and where $\mathfrak{m}_{1/2}$ contains the set of minor arcs $\mathfrak{m}$ defined in §1. It therefore follows from (3.17) that we can find an integer $\tilde{v} \ll tk_1$ for which $2\tilde{v}\sigma(\mathbf{k}) > \Delta_{\tilde{u}}$. By including two copies of $F(\boldsymbol{\alpha})$ and applying a Hardy-Littlewood dissection as in the proof of Lemma 2.2, we find that the estimate (1.5) in fact holds with $\Delta_u = 0$, where $u = \tilde{u} + \tilde{v} + 1$. The first part of the corollary now follows by applying Theorem 1.1 with this value of u and $v = t$.

Finally, if $t \geq \sqrt{k_1}$, then we instead apply Vinogradov's work (see [2], Theorem 4.4), which allows us to take $\sigma(\mathbf{k})^{-1} \sim 8k_1^2 \log k_1$ in (1.6). Then on taking

$$u \sim \tfrac{1}{2}tk_1(3\log k_1 - \log t + 4\log\log k_1),$$

we see from [20], Theorem 2, that (1.5) holds with $\Delta_u \ll t(k_1 \log k_1)^{-1}$. Hence Theorem 1.1 applies directly with $v \ll tk_1$, and the last part of the corollary follows.

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